Energy Equation (Non conservation form)

The physical principle is law of conservation of energy

[Rate of change of energy inside the fluid element] = [Net heat flux into the element] + [Rate of workdone on the element due to body & surface forces] → (1)

$$A = B + C$$

A

Rate of change of energy inside the fluid element

The total energy of a moving fluid per unit mass is the sum of its internal energy per unit mass (e) and its kinetic energy per unit mass ($V^2/2$).

∴ Total energy per unit mass is $e + V^2/2$

$$A = \rho \frac{D}{Dt}\left(e + \frac{V^2}{2}\right) dx\,dy\,dz \longrightarrow (2)$$

B

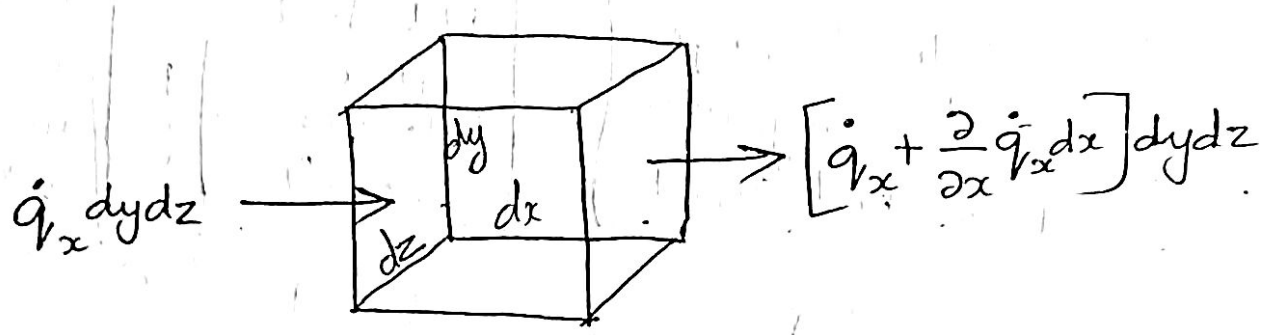
Net flux into the element

The heat flux is due to (i) volumetric heating

(ii) heat transfer across the surface

If $\dot{q}$ is the volumetric heat addition rate per unit mass

Then, volumetric heating of the element $\} = \rho \dot{q}\, dx\, dy\, dz \longrightarrow (3)$



$\dot{q}_x$ is the heat transferred in x-direction per unit time per unit area by conduction.

Net heat transferred in x-direction is

$$= \dot{q}_x dydz - \left[\dot{q}_x + \frac{\partial}{\partial x}\dot{q}_x dx\right] dydz$$

$$= -\frac{\partial}{\partial x}\dot{q}_x\, dx\, dy\, dz$$

Considering y & z direction the heat flux balance is

$$= -\left[\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z}\right] dx\, dy\, dz \longrightarrow (4)$$

The net heat flux

$$B = \left[\rho \dot{q} - \left(\frac{\partial \dot{q}_x}{\partial x} + \frac{\partial \dot{q}_y}{\partial y} + \frac{\partial \dot{q}_z}{\partial z}\right)\right] dx\, dy\, dz \longrightarrow (5)$$

From Fourier law of conduction.

$$\dot{q}_x = -k\frac{\partial T}{\partial x}, \quad \dot{q}_y = -k\frac{\partial T}{\partial y}, \quad \dot{q}_z = -k\frac{\partial T}{\partial z}$$

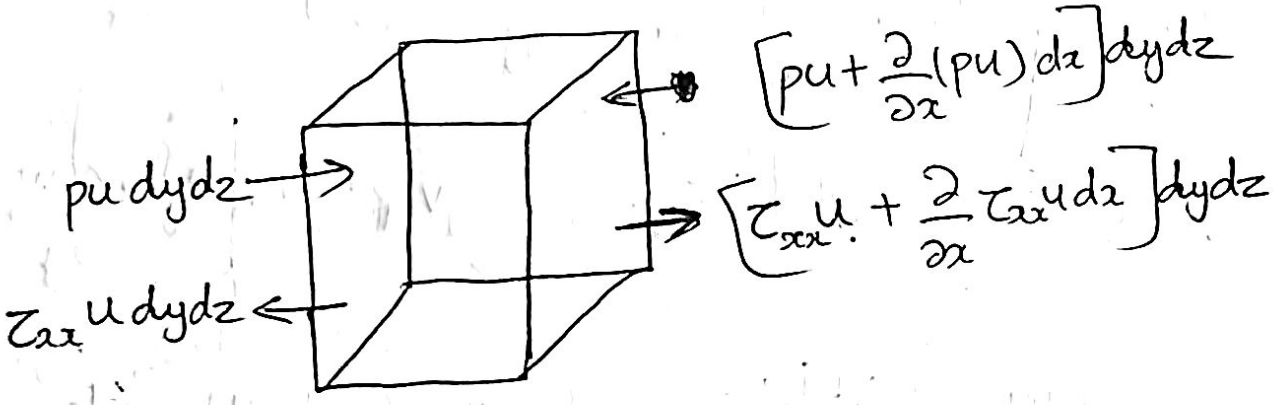
$$\therefore \quad B = \left[\rho\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right] dx\,dy\,dz \longrightarrow (6)$$

C

Rate of workdone on the element due to body & surface forces.

Rate of workdone by body force acting on fluid element moving with velocity $\vec{V}$ is.

$$= \rho \vec{f} \vec{V}\, dx\,dy\,dz \longrightarrow (7)$$



Workdone by pressure force in x direction is

$$= pu\,dy\,dz - \left[pu + \frac{\partial}{\partial x}(pu)\,dx\right]dy\,dz$$

$$= -\frac{\partial}{\partial x}(pu)\,dx\,dy\,dz \longrightarrow (8)$$

Considering all directions (x, y & z)

$$= -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dx\,dy\,dz \longrightarrow (9)$$

Similarly considering workdone by shear and normal stress in all direction.

$$= \left[\frac{\partial}{\partial x}(\tau_{xx}u) + \frac{\partial}{\partial y}(\tau_{yx}u) + \frac{\partial}{\partial z}(\tau_{zx}u) + \right.$$

$$\frac{\partial}{\partial x}(\tau_{xy}v) + \frac{\partial}{\partial y}(\tau_{yy}v) + \frac{\partial}{\partial z}(\tau_{zy}v) +$$

$$\left.\frac{\partial}{\partial x}(\tau_{xz}w) + \frac{\partial}{\partial y}(\tau_{yz}w) + \frac{\partial}{\partial z}(\tau_{zz}w)\right] dx\,dy\,dz$$

$\longrightarrow (10)$

Now total workdone by all forces adding eq (7), (9) & (10)

$$C = \left[\rho \vec{f}\,\vec{V} - \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) + \frac{\partial \tau_{xx}u}{\partial x} + \frac{\partial \tau_{yx}u}{\partial y} + \frac{\partial \tau_{zx}u}{\partial z} + \right.$$

$$\left.\frac{\partial \tau_{xy}v}{\partial x} + \frac{\partial \tau_{yy}v}{\partial y} + \frac{\partial \tau_{zy}v}{\partial z} + \frac{\partial \tau_{xz}w}{\partial x} + \frac{\partial \tau_{yz}w}{\partial y} + \frac{\partial \tau_{zz}w}{\partial z}\right] dx\,dy\,dz$$

$\longrightarrow (11)$

Now substituting eq (2), (6) & (11) into eq (1)

$$A = B + C$$

Soure flow

Consider a flow with straight streamline originating from a point, where velocity along each streamline varies inversely with distance from the point, such a flow is called source flow. This flow is irrotational

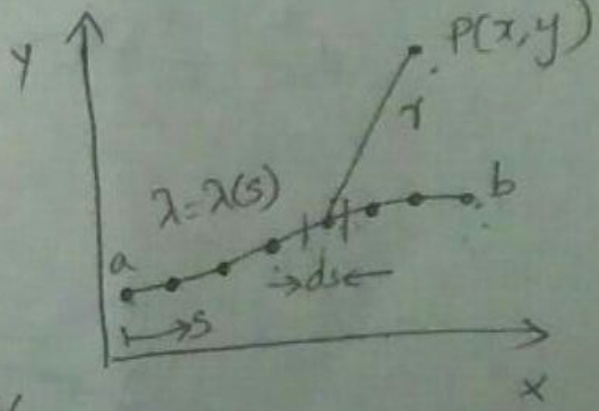
$$\phi = \frac{\Lambda}{2\pi} \ln r \qquad \Lambda \; \lambda$$

where Λ is the source strength and is the rate of volume flow from source per unit depth perpendicular to the page or plane.

Non-lifting flow over arbitrary body – Source panel method

Consider a single soure sheet as shown which composes of several line sources.

Let $\lambda = \lambda(s)$ be the source strength per unit length along 's' direction



Then λ is expressed in m^2/s

The strength of infinitesimal portion ds is λds.

Now consider point P(x,y) at distance r from ds

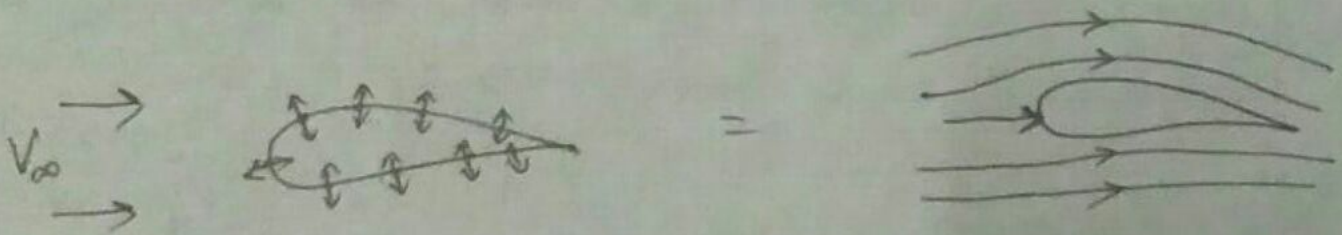
This small section of source sheet of strength λds induces an infinitesimal small potential $d\phi$ at point P.

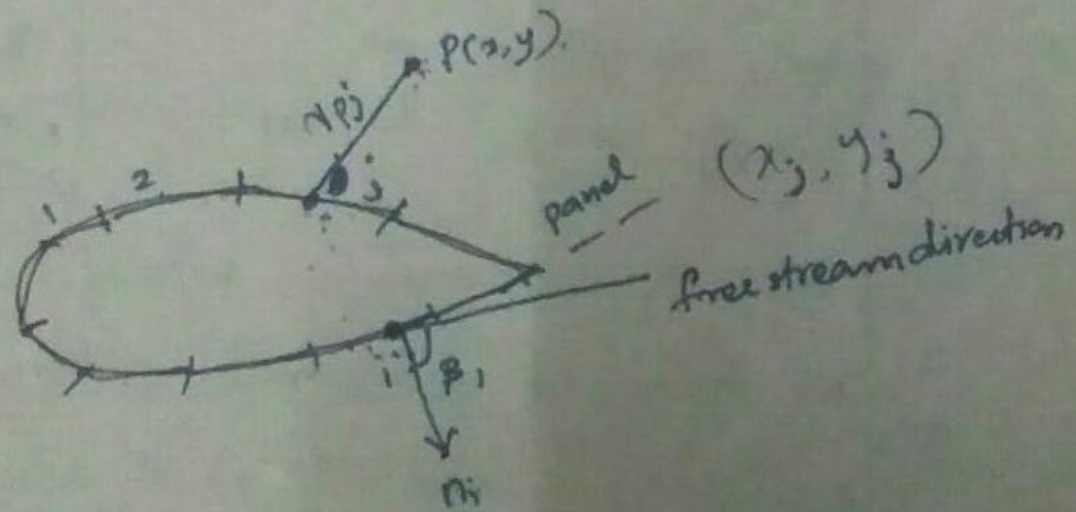
$$d\phi = \frac{\lambda ds}{2\pi} \ln r \longrightarrow (1)$$

The complete velocity potential at P induced by the entire source sheet from a to b is

$$\phi(x,y) = \int_a^b \frac{\lambda ds}{2\pi} \ln r \longrightarrow (2)$$

Consider a given body of arbitrary shape in a flow with free stream velocity V_∞. Let us approximate the source sheet by a series of ~~pol~~ straight panels. The source strength λ per unit length is constant over a panel but varies from one panel to another.





The strength of each panel per unit length is $\lambda_1, \lambda_2 \ldots \lambda_j \ldots \lambda_i \ldots \lambda_n$, where j varies from 1 to n. The concept of panel technique is to solve for λ_j, such that the body surface becomes streamline of flow. This boundary condition is imposed numerically by defining midpoint of each panel to be the 'control point' such that the normal component of the flow velocity is zero at each control point.

The velocity potential induced at P due to the j^{th} panel.

$$d\phi_j = \frac{\lambda_j}{2\pi} \int_j \ln r_{Pj} \, ds_j \longrightarrow (3)$$

λ_j is constant over j^{th} panel

The potential at P due to all panels is

$$\phi(P) = \sum_{j=1}^{n} d\phi_j = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{Pj} \, ds_j \longrightarrow (4)$$

the distance $r_{Pj} = \sqrt{(x - x_j)^2 + (y - y_j)^2}$ $\longrightarrow (5)$

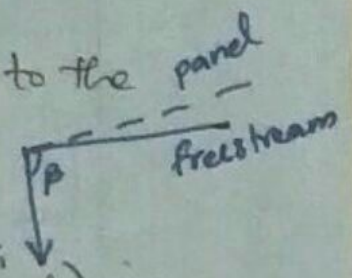
(x_j, y_j) is the coordinates of j^{th} panel

Now let us assume the Point 'P' to be at the control point of i^{th} panel with coordinates (x_i, y_i)

$$\phi(x_i, y_i) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{ij} \, ds_j \longrightarrow (6)$$

where $r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ $\longrightarrow (7)$

Equation (6) is the contribution of all panels to the potential at the control point of i^{th} panel.



The normal component of V_∞ (normal to i^{th} panel) is

$$V_{\infty,n} = V_\infty \cdot \vec{n}_i = V_\infty \cos\beta_i \longrightarrow (8)$$

The normal component of velocity induced at (x_i, y_i) by source panel is

$$V_n = \frac{\partial}{\partial n_i} [\phi(x_i, y_i)] \longrightarrow (9)$$

while solving a singular point arises on i^{th} panel, when $j = i$ at the control point $r_{ij} = 0$. It can be seen when $j = i$ the contribution to derivative is $\lambda_i/2$. Hence

$$V_n = \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} (\ln r_{ij}) \, ds_j \longrightarrow (10)$$

From boundary condition

$$V_{\infty,n} + V_n = 0 \longrightarrow (11)$$

$$\therefore \quad \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i} (\ln r_{ij}) \, ds_j + V_\infty \cos\beta_i = 0 \longrightarrow (12)$$

The velocity tangent to the surface at each control point is as follows.

The component of free stream velocity tangent to surface is

$$V_{\infty,s} = V_\infty \sin\beta_i \longrightarrow (13)$$

The tangential velocity V_s at the control point of the i^{th} panel induced by all panels is

$$V_s = \frac{\partial\phi}{\partial s} = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial s}(\ln r_{ij})\, ds_j \longrightarrow (14)$$

The tangential velocity on a flat source panel induced by the panel itself is zero Hence in the above equation the term corresponding to $j=i$ is zero.

The total velocity now at i^{th} control point

$$V_i = V_{\infty,s} + V_s = V_\infty \sin\beta_i + \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial s} \ln(r_{ij})\, ds_j \longrightarrow (15)$$

By applying Bernoulli's theorem the coefficient of pressure at i^{th} control point is

$$C_{p,i} = 1 - \left(\frac{V_i}{V_\infty}\right)^2$$

Thus the source panel method gives the pressure distribution over a surface of non-lifting body.

The governing eqns of fluid mechanics are expressed as partial differential equations. Second order partial diff. eqns appear frequently and therefore are of particular interest in fluid mechanics and heat transfer. Generally the governing eqns of fluid mechanics form a set of coupled, non-linear p.d.es which must be solved in an irregular domain subjected to various initial and boundary conditions.

The fundamental concept of numerical scheme is based on the approximation of partial derivatives by simple algebric expressions. Once the P.D.E is approximated by algebric equations, it can be solved numerically with the help of a computer. The schemes by which the approximations to the P.D.E can be develop may be categoired in to three groups.

1) FDM (Finite Difference Method)
2) FEM (Finite Element Method)
3) FVM (Finite Volume Method)

The solution of P.D.E's depends on the type of eqn, so it is ~~involved at the~~ important to study the various classifications of P.D.E's. The application of initial or boundary condition also depends on the type of P.D.E.

Most of the governing eqn in fluid mechanics and heat transfer are expressed as second order PDEs.

The PDE can be classified as linear or nonlinear equations. In a linear eqn, the dependent variable and its derivative appear in the eqn in linear form. i'e; there is no product of the dependent variable or its derivatives.

Ex: One dimensional wave equation.

$$\text{i.e., } \frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}$$

The non-linear PDE contains a product of the dependent variable or the product of derivatives or a combination of both. eg: Inviscid Burger's eqn.

$$\text{i.e., } \frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x}$$

If a PDE is linear in its highest order derivative, then it is referred as quasi-linear PDE.

Finite Difference Method

In order to solve a given PDE, the partial derivatives in the eqn are approximated by finite difference relations. These approximations of the partial derivatives are obtained from Taylor series expressions. The resulting approximate eqn, which represents the given PDE is referred as finite difference eqn (FDE).

Finite Element Technique

The finite element method is a numerical technique for solving partial diff eqns. Its first essential characteristic is that the continuum field or the domain is subdivided in to small portions which are referred as elements. The elements have a triangular or quadrilateral form and can be rectilinear or curved. With the help of unstructured grid structure, any complex geometries can be easily handled.

The second essential characteristic of the finite element method is that the solution of the discrete problem is assumed to have a prescribed form, the solution has to belong to a function in space. The nodes are the typical points of the elements such as vertices, mid-side points or mid-element points. The representation of the solution is strongly linked to the geometric representation of the domain.

The third essential characteristic is that the finite element method doesnot look for the solution of the PDE but looks for a solution of an integral

form of PDE. The most general form is obtained from a weighted residual formulation. By this formulation, the method acquires the ability to naturally incorporate the different types of boundary condition.

A final essential characteristic of the finite element method is the modular way in which the discretization is obtained. The discrete equations are obtained from the contribution on the element level which are then finally assembled.

Mathematical properties of fluid dynamics equation

For an equation

$$a\left(dy/dx\right)^2 + b\left(dy/dx\right) + c = 0.$$

The solution is

$$\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

* If $b^2 - 4ac > 0$, two real and distinct characteristics exist through each point. This equation is hyperbolic.

* If $b^2 - 4ac = 0$, One real characteristic exists. The equation is parabolic.

* If $b^2 - 4ac < 0$, The characteristics lines are imaginary. The equation is elliptic.

Hyperbolic equation

A PDE is hyperbolic if $b^2 - 4ac > 0$ at all points of the region. It has two real characteristics.

eg: First order wave equation → $\partial\phi/\partial t = a\,\partial\phi/\partial x$

Second order wave equation → $\frac{\partial^2\phi}{\partial t^2} = a^2\frac{\partial^2\phi}{\partial x^2}$

Parabolic equation

A PDE is classified as parabolic if $b^2 - 4ac = 0$ at all points of the region. The solution of the domain is an open region and there exist one characteristic.

eg:- unsteady heat conduction equation (1D)

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2}$$

Elliptic equation

A PDE is elliptic in a region if $b^2 - 4ac < 0$ at all points of the region. An elliptic PDE has no real characteristic curves.

eg: Laplace equation → $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

Poisson's equation → $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x,y)$

The domain of solution for an elliptic PDE is a closed region.

Initial and boundary conditions :-

In order to obtain a unique solution of PDE, a set of supplementary conditions must be provided. These supplementary conditions are classified as, boundary and initial conditions.

An initial condition is a requirement for which the dependent variable is specified at an initial state ($t=0$).

A boundary condition is a requirement that the dependent variable or its derivative must satisfy on the boundary of the domain of PDE.

Types of boundary conditions

1) Dirichlet boundary condition :-

If the dependent variable is specified around the boundary, then it is referred as dirichlet boundary condition. (eg: u)

2) Neumann boundary condition :-

If the normal gradient of the dependent variable is specified along the boundary, then it is referred as Neumann boundary condition. (eg: $\frac{\partial u}{\partial x}$)

3) Mixed boundary condition :-

Fuquently, the boundary condition along a certain part of the boundary is Dirichlet type and on other part, it is of Neumann type and this case is referred as mixed boundary condition.

Strong Formulation

Consider as an example, the following simple one-dimensional boundary value problem given by the form,

$$\frac{d}{dx}\left(\lambda \frac{du}{dx}\right) = f \longrightarrow (1)$$

on $0 \leq x \leq X$

The boundary condition is

$$u(0) = u_0 \longrightarrow (2)$$

$$\lambda \frac{du}{dx}(X) = q \longrightarrow (3)$$

Now generally, the diff eqn [eq (1)] can be written as,

$$a(u) = f \quad \text{——} (4)$$

The domain to which the above eqn is applied is denoted by Ω: omega (cap)

The boundary conditions given by eqn (2) is referred as Dirichlet boundary condition. It can be generally expressed as, $b_0(u) = g_0$ —— (5)

The boundary condition given by eqn (3) is referred as Neumann boundary condition and it can be generally expressed as,

$$b_1(u) = g_1 \quad \text{——} (6)$$

The boundary of the domain is denoted by 'Γ', the part to which the Dirichlet boundary condition is applied is denoted by, 'Γ_0' and the part to which the Neumann boundary condition is applied is denoted by 'Γ_1'.

The boundary value problem given by eqn (1), (2) and (3) is said to be in <u>strong form</u>, if the differential eqn given by eqn (1) is satisfied at all points of the

domain Ω, the Dirichlet boundary eqn given by eqn ② is satisfied at all points of the boundary 'Γ_0' and the Neumann boundary condition is given by eqn ③ is satisfied is all points of the boundary 'Γ_1'.

Weighted residual formulation

The first basic ingredient of the finite element method is that an approximate solution which belongs to some function in space is considered. The approximate solution of the boundary value problem given by eqn ①, ② and ③ has the form, given by the expression,

$$\hat{u} = \psi + \sum_{k=1}^{n} \phi_k u_k \qquad \text{——————} ④$$

where ψ is the function which satisfies the boundary condition represented by eqn ② and ③, the function ϕ_k are referred as shape function or geometric function, where $k = 1, 2, 3 \ldots n$. The approximate solution given by eqn ④ cannot satisfy the differential equation represented by eqn ① in each point of the domain. This means that the approximate solution ($\hat{u}$) cannot be identical with the exact solution (u). The shape function should be chosen so that, by

enriching a function in space (ϕ) ie; by allowing n to grow the approximation obtained from eqn ② becomes better. This means that the approximate solution converges to exact solution. This is referred as the completeness requirement of a function in space.

Since the approximate solution given by eqn ② cannot satisfy the differential equation represented by eqn ①, a residual is left over which is given by the expression,

$$R_\Omega = a(\hat{u}) - f \quad \text{——— ⑧}$$

An approximate solution of the boundary value problem is now obtained by finding a way to minimize this residual. In FEM, this is achieved by requiring that an appropriate no. of weighted integrals of the residual over the domain is equated to zero and is represented as follows;

$$\int_\Omega w_i R_\Omega \, d\Omega = 0 \quad \text{——— ⑨}$$

$w_i \rightarrow$ weighting function

$i = 1, 2, 3 \ldots n$

The above eqn now implies a requirement of minimization of the residue and can be expressed as follows.

when $n \longrightarrow \infty$

$R_n \longrightarrow 0$

Galerkin Formulation :-

Among the possible choices of weighting function, the following are the most commonly used types.

1) The weighting function can be chosen to be a dirac-delta function at 'n' number of points. This choice means approaching the residual to zero in a no. of chosen points. This method is referred as point-collocation method.

2) A second possible choice of the weighting function can be represented by the eqn as follows;

$W_i = 1$ for $x_i \leq x \leq x_{i+1}$

$W_i = 0$ for $x < x_i, x > x_{i+1}$

The above method is referred as subdomain-collocation method.

3) Galerkin method :-

The most popular choice for the waiting function in the FEM is the shape function itself. In this method, the waiting function can be chosen as its shape function.

$$w_i = \phi_i$$

It means that the residual is made to be orthogonal to a function in space. To illustrate a Galerkin method, consider a boundary value problem given by the expression,

$$\frac{d}{dx}\left(\lambda \frac{du}{dx}\right) = f$$

$$u(0) = u_0$$

$$\lambda \frac{du}{dx}(x) = q$$

Since ψ is a function which satisfies the boundary condition. It can be given by the expression,

$$\psi = u_0 + \frac{q}{\lambda} x \quad \text{——— (10)}$$

From Dirichlet boundary condition
$u = u_0$

From Neumann
$du = \frac{q}{\lambda} dx$

ie $u = \frac{q}{\lambda} x$

Now consides further an example of the approximate solution (eqn ⑦), a fourier sine expansion of u

$$\hat{u} = \psi + \sum_{k=1}^{N} u_k \sin\left(\frac{\pi k x}{X}\right) \quad \text{——— (11)}$$

The residue can be given by the expression,

$$r_\Omega = a(\hat{u}) - f$$

$$r_\Omega = \frac{d}{dx}\left(\lambda \frac{d\hat{u}}{dx}\right) - f$$

$$r_\Omega = \frac{d}{dx}\left(\lambda \frac{d}{dx}\left[\psi + \sum_{k=1}^{N} u_k \sin\left(\frac{\pi k x}{X}\right)\right]\right) - f$$

∴ The expression becomes,

From ⑥

$$\frac{d\psi}{dx} = \frac{q}{\lambda} \; ; \; \frac{d^2\psi}{dx^2} = 0$$

$$r_\Omega = -\lambda \sum_{k=1}^{N} u_k \left(\frac{\pi k}{X}\right)^2 \cdot \sin\left(\frac{\pi k x}{X}\right) - f \quad \text{——— (12)}$$

By Galerkin method we can write,

Since $\frac{d}{dx}(\sin mx) = m\cos mx$

and $\frac{d}{dx}(\cos mx) = -m\sin mx$

$$\int_\Omega w_i r_\Omega \, d\Omega = 0$$

The Galerkin method can be expressed as,

$$\int_0^X \sin\frac{\pi i x}{X}\left[-\lambda\sum_{k=1}^{N} u_k\left(\frac{\pi k}{X}\right)^2 \sin\left(\frac{\pi k x}{X}\right) - f\right] dx = 0$$

4) A fourth weighted residual method exists in FEM in which the formulation is based on minimization of the integral and is referred as <u>least square method</u>.

$$\int_{\Omega} r_{\Omega}^2 \, d\Omega$$

Weak Formulation

In many problems it is not practical to construct a function which satisfies the boundary conditions in order to arrive at an expression for the approximate solution. The approximate solution can be expressed as

$$\hat{u} = \sum_{k=1}^{n} \phi_k u_k \longrightarrow (1)$$

Now the approximate solution not only has a residual with respect to the differential equation but also with respect to the boundary condition equations.

$$r_0 = b_0(\hat{u}) - b_0(u) \longrightarrow (2)$$

$$r_0 = b_0(\hat{u}) - g_0$$

$$r_0 = \hat{u}(0) - u(0)$$

$$r_0 = \hat{u}(0) - u_0 \longrightarrow (3)$$

$$r_1 = b_1(\hat{u}) - b_1(u) \longrightarrow (4)$$

$$r_1 = b_1(\hat{u}) - g_1$$

$$r_1 = \lambda \frac{d\hat{u}}{dx}(x) - \lambda \frac{du}{dx}(x) = \lambda \frac{d\hat{u}}{dx}(x) - q$$

$\longrightarrow (5)$

Now the weighted residual statement can be expressed as

$$\int_{\Omega} w_i r_\Omega \, d\Omega + \int_{\Gamma_0} w_i^0 r_0 \, d\Gamma + \int_{\Gamma_1} w_i^1 r_1 \, d\Gamma = 0. \quad \rightarrow (6)$$

The weak formulation is complicated because of additional weighting function on the boundaries.

Considering the limits the the residual can be substituted with respect to boundary from eqn (3) & (5). Then

$$\int_0^X w_i \left[\frac{d}{dx}\left(\lambda \frac{d\hat{u}}{dx}\right) - f\right] dx + w_i^0 \left[\hat{u}(0) - u_0\right] + w_i^1 \left[\lambda \frac{d\hat{u}}{dx}(x) - q\right] = 0 \quad \rightarrow (7)$$

Since integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\left[w_i \lambda \frac{d\hat{u}}{dx}\right]_0^X - \int_0^X \lambda \frac{d\hat{u}}{dx} \frac{dw_i}{dx} dx - \int_0^X w_i f \, dx + w_i^0 \left[\hat{u}(0) - u_0\right] + w_i^1 \left[\lambda \frac{d\hat{u}}{dx}(x) - q\right] = 0 \quad \rightarrow (8)$$

This weighted residual statement is simplified by choosing the weighting factors on Neumann boundary by

$$w_i^1 = -w_i(x)$$

Now equation becomes.

$$\left[w_i \lambda \frac{d\hat{u}}{dx}\right]_0^x - \int_0^x \lambda \frac{d\hat{u}}{dx}\frac{dw_i}{dx}\,dx - \int_0^x w_i f\,dx + w_i^0\left[\hat{u}(0) - u_0\right] - w_i(x)\left[\lambda \frac{d\hat{u}}{dx}(x) - q\right] = 0$$

$$w_i(x)\lambda\frac{d\hat{u}}{dx}(x) - w_i(0)\lambda\frac{d\hat{u}}{dx}(0) - \int_0^x \lambda\frac{d\hat{u}}{dx}\frac{dw_i}{dx}\,dx - \int_0^x w_i f\,dx - w_i^0\left[\hat{u}(0) - u_0\right] + w_i(x)\lambda\frac{d\hat{u}}{dx}(x) + w_i(x)q = 0$$

$$- w_i(0)\lambda\frac{d\hat{u}}{dx}(0) - \int_0^x \lambda\frac{d\hat{u}}{dx}\frac{dw_i}{dx}\,dx - \int_0^x w_i f\,dx - w_i^0\left[\hat{u}(0) - u_0\right] + w_i(x)q = 0$$

Further if the Dirichlet boundary condition can be imposed on the approximate solution the weighting function can be chosen to be zero at dirichlet boundary so the equation becomes.

$$\hat{u}(0) = u_0$$

$$w_i(0) = 0.$$

$$-\int_0^x \lambda \frac{d\hat{u}}{dx} \frac{dw_i}{dx} dx - \int_0^x w_i f dx + w_i(x) q = 0.$$

The above weighted residual statement is referred as <u>weak formulation</u>

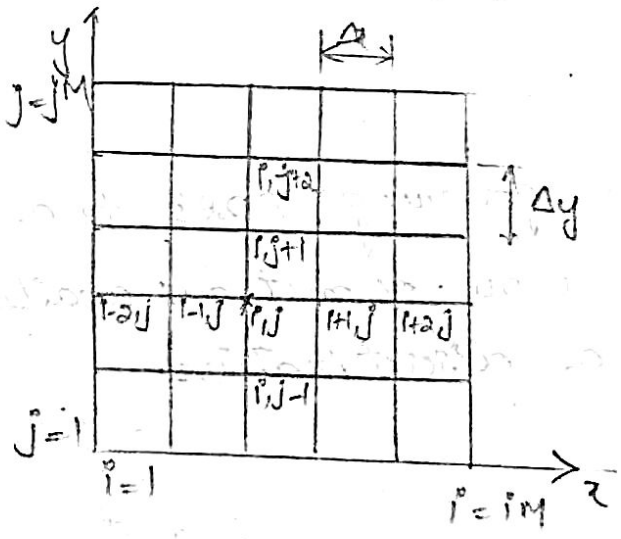
Discretization

In order to convert the governing PDE's to a s/m of algebric eqns, there are a no. of methods available which are collectively referred as discretization.

Finite Difference Method

In order to solve a given PDE, the partial derivatives in the eqn are approximated by finite difference relations. These approximations of the partial derivatives are obtained from Taylor series expressions. The resulting approximate eqn, which represents the original PDE is referred as finite difference eqn (FDE).

Consides a 2-D rectangular domain as shown in the figure. A PDE is to be solved within the domain subjected to initial and boundary conditions. The rectangular domain is divided in to increments in x and y direction. These increments may be defined as mesh size, grid size or step size. The location of the mesh or grid points or nodes is denoted by i in x-direction and j in y-direction.



i, j → Reference point

Taylor Series Expansion

Let $f(x)$ be a function. Then using Taylor series expansion we can write,

$$f(x+\Delta x) = f(x) + \Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \cdots \quad \text{——— (1)}$$

$$f(x-\Delta x) = f(x) - \Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \cdots \quad \text{——— (2)}$$

From eqn (1) we can write

$$\Delta x \frac{\partial f}{\partial x} = f(x+\Delta x) - f(x) - \frac{(\Delta x)^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{(\Delta x)^3}{6}\frac{\partial^3 f}{\partial x^3}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2} - \frac{(\Delta x)^2}{6}\frac{\partial^3 f}{\partial x^3}$$

i.e. $$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x)$$

↓ order of Δx

The $O(\Delta x)$ is also referred as Truncation error.

If the alphabet 'i' is used to represent the discrete

points in x-direction. The above eqn can be written as

$$\boxed{\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x)} \quad \text{——— (3)}$$

The above eqn is referred as forward difference approximation or formulation of the 1st order derivative $\left(\frac{\partial f}{\partial x}\right)$ and is of order Δx.

From eqn (2) we can write,

$$\frac{\partial f}{\partial x} = \frac{f(x) - f(x-\Delta x)}{\Delta x} + \frac{\Delta x}{2}\frac{\partial^2 f}{\partial x^2} - \frac{(\Delta x)^2}{6}\frac{\partial^3 f}{\partial x^3}$$

ie $$\boxed{\frac{\partial f}{\partial x} = \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x)} \quad \text{——— (4)}$$

The above eqn is the backward difference approximation or formulation of the 1st order derivative $\left(\frac{\partial f}{\partial x}\right)$ and is of order Δx.

The order is also expressed in terms of accuracy. Δx is also called as first order accurate.

Now subtracting eqn ② from eqn ①,

① − ② ⇒

$$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x \frac{\partial f}{\partial x} + 2\frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \cdots$$

i.e $$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^3}{3}\frac{\partial^3 f}{\partial x^3} + \cdots$$

$$\therefore \frac{\partial f}{\partial x} = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{(\Delta x)^2}{6}\frac{\partial^3 f}{\partial x^3}$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x)^2} \quad \text{——— ⑤}$$

The above eqn is the central difference approximation or formulation of the 1st order derivative $\left(\frac{\partial f}{\partial x}\right)$ and is of order $(\Delta x)^2$ i.e second order accurate.

Higher Order Derivatives

Consider the Taylor series expression,

$$f(x+2\Delta x) = f(x) + 2\Delta x \frac{\partial f}{\partial x} + \frac{(2\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{(2\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{(2\Delta x)^4}{4!}\frac{\partial^4 f}{\partial x^4} + \cdots \quad \text{——— ①}$$

$$f(x-2\Delta x) = f(x) - 2\Delta x\frac{\partial f}{\partial x} + \frac{(2\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} - \frac{(2\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{(2\Delta x)^4}{4!}\frac{\partial^4 f}{\partial x^4} + \cdots$$

$$f(x+\Delta x) = f(x) + \Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 f}{\partial x^4} + \ldots$$

—— (2)

Now by multiplying eqn (2) by 2 and subtracting from eqn (1) we get,

$$f(x+2\Delta x) - 2f(x+\Delta x) = -f(x) + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = +f(x) + f(x+2\Delta x) - 2f(x+\Delta x) - \Delta x \frac{\partial^3 f}{\partial x^3}$$
$$(\Delta x)^2$$

ie $$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{f_i - 2f_{i+1} + f_{i+2}}{(\Delta x)^2} + O(\Delta x)}$$

The above eqn is the forward difference approximation or formulation of 2nd order derivative $\left(\frac{\partial^2 f}{\partial x^2}\right)$ and is of order Δx.

Similarly the backward difference formulation can be obtained from the expression,

$$f(x-2\Delta x) = f(x) - 2\Delta x \frac{\partial f}{\partial x} + \frac{(2\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} - \frac{(2\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{(2\Delta x)^4}{4!}\frac{\partial^4 f}{\partial x^4} - \ldots$$

—— (3)

$$f(x-\Delta x) = f(x) - \Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^3} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 f}{\partial x^4} - \cdots \quad \text{——} (5)$$

(3) − 2 × (4) ⟹

$$f(x-2\Delta x) - 2f(x-\Delta x) = -f(x) + \frac{2(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^3} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{f(x) + f(x-2\Delta x) - 2f(x-\Delta x)}{(\Delta x)^2} + \Delta x \frac{\partial^3 f}{\partial x^3}$$

i) $$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{f_p - 2f_{p-1} + f_{p-2}}{(\Delta x)^2} + O(\Delta x)}$$

ii) $$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{f_{p-2} - 2f_{p-1} + f_p}{(\Delta x)^2} + O(\Delta x)}$$

The above eqn is the backward diff formulation of approximation of 2nd order derivative $\left(\frac{\partial^2 f}{\partial x^2}\right)$ and is of order Δx

We have,

$$f(x+\Delta x) = f(x) + \Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 f}{\partial x^4} \quad \text{——— (5)}$$

$$f(x-\Delta x) = f(x) - \Delta x \frac{\partial f}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{(\Delta x)^4}{4!}\frac{\partial^4 f}{\partial x^4} \quad \text{——— (6)}$$

(5) + (6) $\Rightarrow$

$$f(x+\Delta x) + f(x-\Delta x) - 2f(x) + \frac{2(\Delta x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + \frac{2(\Delta x)^4}{4!}\frac{\partial^4 f}{\partial x^4}$$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{-2f(x) + f(x+\Delta x) + f(x-\Delta x)}{(\Delta x)^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 f}{\partial x^4}$$

i.e.

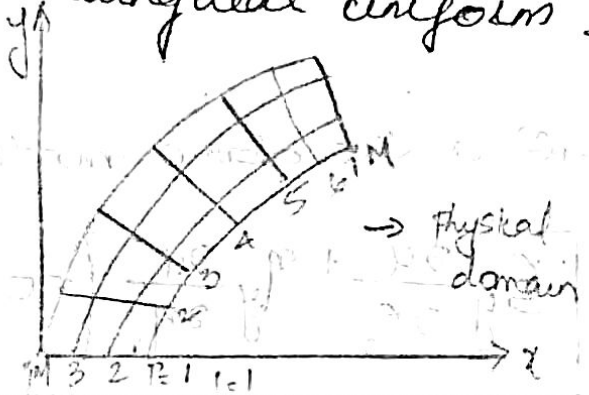
$$\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{f_{i-1} - 2f_i + f_{i+1}}{(\Delta x)^2} - O(\Delta x)^2}$$

The above eqn is the central difference approximation or formulation of the 2nd order derivative $\left(\frac{\partial^2 f}{\partial x^2}\right)$ and is of order $(\Delta x)^2$. or 2nd order accurate.

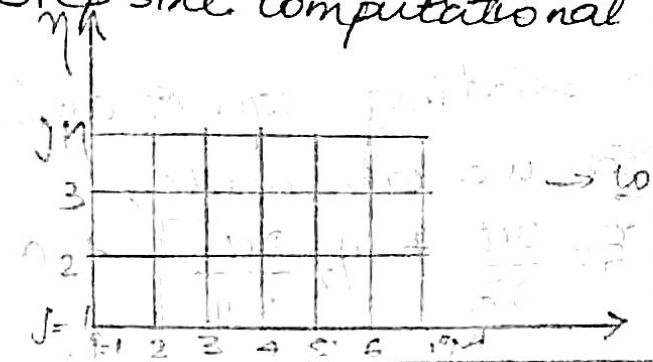
Transformation

The generation of grid with uniform spacing is simple within a rectangular domain, unfortunately the majority of the physical domains are not rectangular. Therefore imposing a rectangular computational domain on a physical domain will require some set of interpolations for the implementation of boundary conditions.

The transformation from physical space to the computational space is accomplished by specifying a generalised co-ordinate system which will map the non-rectangular grid elements in the physical plane to a rectangular uniform grid spacing in the computational plane. In order to eliminate the difficulty associated with the non-equal step sizes used in the finite difference equations, the physical domain is transformed in to a rectangular uniform step size computational domain.



→ Physical domain



→ Computational domain

The co-ordinate transformation can be explained as follows.

$$\xi = \xi(x, y) \quad \text{——} ①$$

$$\eta = \eta(x, y) \quad \text{——} ②$$

By chain rule,

Generally $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} \quad \text{——} ③$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} \quad \text{——} ④$$

The above eqns can be re-arranged as,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \cdot \xi_x + \frac{\partial f}{\partial \eta} \eta_x \quad \text{——} ⑤$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi} \xi_y + \frac{\partial f}{\partial \eta} \cdot \eta_y \quad \text{——} ⑥$$

Consider a model equation as written below,

$$\frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y} = 0$$

Now, relating eqn ⑤ and ⑥ to the above model equation, we can write,

$$\left[\xi_x \frac{\partial u}{\partial \xi} + \eta_x \frac{\partial u}{\partial \eta}\right] + a \left[\xi_y \frac{\partial u}{\partial \xi} + \eta_y \frac{\partial u}{\partial \eta}\right] = 0$$

i.e., $\frac{\partial u}{\partial \xi}\left(\xi_x + a\xi_y\right) + \frac{\partial u}{\partial \eta}\left(\eta_x + a\eta_y\right) = 0$

where $\xi_x, \xi_y, \eta_x \& \eta_y$ should be determined from the metrics.

From eqn ① and ② we can write,

$d\xi = \xi_x dx + \xi_y dy$ [from $d\xi = \frac{d\xi}{dx}dx + \frac{d\xi}{dy}dy$]

$d\eta = \eta_x dx + \eta_y dy$ [from $d\eta = \frac{d\eta}{dx}dx + \frac{d\eta}{dy}dy$]

The above equations can be expressed as,

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix} \quad \text{——— ⑦}$$

Now reversing the role of independent variables, we can write,

$x = x(\xi, \eta)$ ——— ⑧

$y = y(\xi, \eta)$ ——— ⑨

$dx = x_\xi d\xi + x_\eta d\eta$

$dy = y_\xi d\xi + y_\eta d\eta$

$$\therefore \begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} \quad \text{——— ⑩}$$

Now on comparing eqn ⑦ and ⑩, we can write,

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix}^{-1}$$

$$\therefore \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = \frac{\begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix}}{(x_\xi y_\eta - y_\xi x_\eta)}$$

Well-posed problem

The governing eqns and its auxillary conditions (ie., initial and boundary conditions) are said to be well-posed if the following three conditions are satisfied.

⇒ The solution exist.

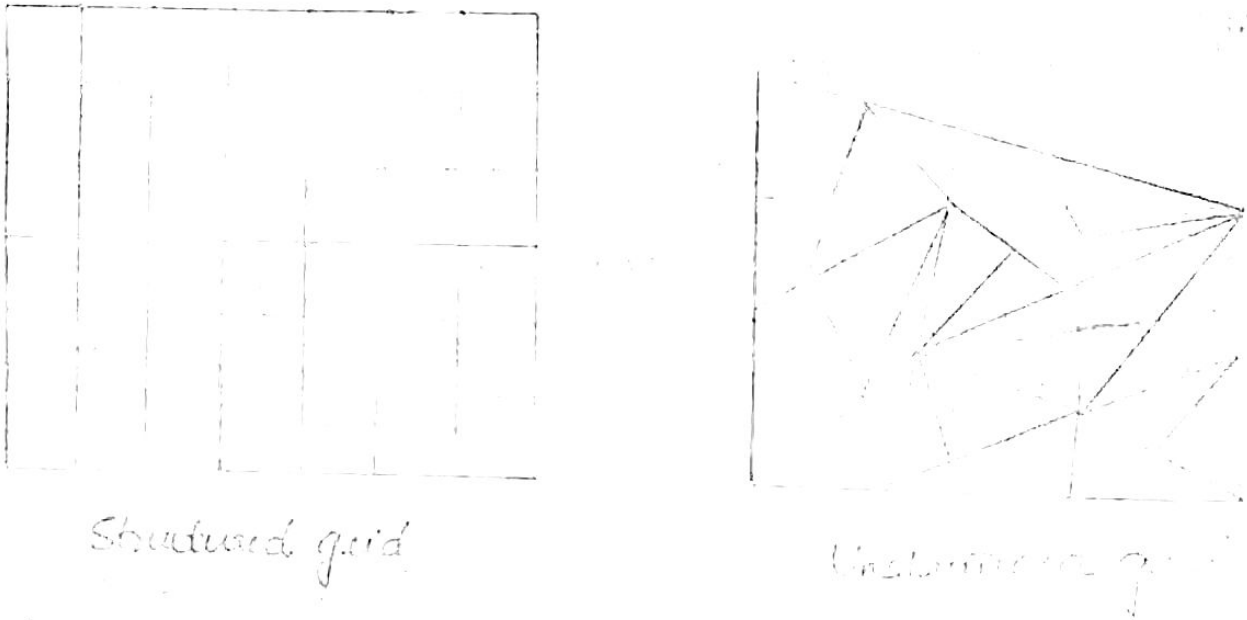
⇒ The solution is unique.

⇒ The solution depends continuously on the initial and boundary condition.

Grid generation :-

Inorder to numerically solve the governing PDE of fluid mechanics, the approximations to PDE's are introduced. These approximations convert the PDE to finite difference eqns. These eqns are then solved at discrete points within the domain of interest. Therefore, a set of grid points within the domain as well as the boundaries of the domain must be specified. Typically, the computational domain is selected to be rectangular in shape where the interior grid points are distributed along the grid lines. This type of grid is referred as structured grid. A second category or grid system

may be constructed where the grid points cannot be associated with orderly defined grid lines and this type of grid is referred as <u>unstructured grid</u>.



Structured grid

In determining the grid points, a few constraints must be imposed.

1) The mapping should be one-to-one (1–1) ie; the grid lines of the same family should not cross each other.
2) A smooth grid point distribution with a minimum grid line skewness.
3) Orthogonality or near orthogonality must be achieved.
4) Options for grid point clustering ie; concentration of grid points in regions where high flow gradients occur.

In general, the grid generation techniques may be classified as,

(i) Algebric method

(ii) Partial differential method

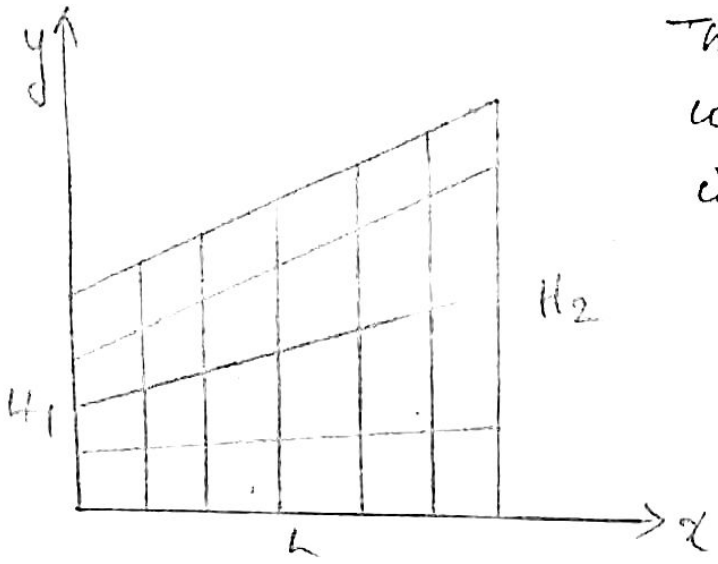
(iii) Conformal mapping.

In addition, the grid system may be also classified as fixed or adaptive. A fixed grid system is generated prior to the solution of the governing equation of fluid motion and remains fixed indipendent of the solution.

On the other hand, the adaptive grid system evolves as a result of the solution of the equations of fluid motion.

Algebric grid generation technique

The simplest grid generation technique is the algebric method in which the grids can be generated easily and the advantage of this method is the speed with which the grids can be generated. An algebric eqn is used to relate the grid points in the computational domain to those of the physical domain. This objective is achieved by using an interpolation scheme to generate the interior grid points.



The physical domain in cartesian coordinates (x, y) is transformed in to computational plane (ξ, η)

The following relations will transform the non-rectangular physical domain to a rectangular computational domain.

$$\xi = x \quad \text{——} \textcircled{1}$$

$$\eta = \frac{y}{y_t} \quad \text{——} \textcircled{2}$$

y_t represents upper boundary.

$$y_t = H_1 + \left(\frac{H_2 - H_1}{L}\right) x \quad \text{——} \textcircled{3}$$

$$\left.\begin{aligned} \xi &= x \\ \eta &= \frac{y}{H_1 + \left(\frac{H_2 - H_1}{L}\right) x} \end{aligned}\right\} \textcircled{4}$$

which can be rearranged as,

$$x = \xi$$
$$y = \left[H_1 + \left(\frac{H_2 - H_1}{L}\right)\xi\right]\eta \quad \Big\} \quad (5)$$

The equal grid spacing in the computational domain has to be achieved.

Advantages :-

1) Computationally very fast.
2) This method provides the ability to cluster the grid points.

Disadvantages:-

1) The discontinuity at the boundary may propogate in to the interior domains.
2) Controlling the smooth distribution of grid points is a difficult task.

Partial differential equation techniques

In this method, a s/m of partial diff eqn is solved for the location of the grid points in the physical space. whereas the computational domain is rectangular in shape with uniform grid spacing. This technique can

be classified as elliptic, parabolic or hyperbolic.

Elliptic grid generators :-

For domains where all the physical boundaries are specified the elliptic grid generators are used. A sys of elliptic eqns in the form of Laplace or Poissons eqn is solved in the grid points of the domain Any iterative schemes such as Gauss Siedel method may be used to analyse the problem.

Advantages :-

1) This method provides a smooth grid point distribution, i.e., if a boundary discontinuity exit, it will be smoothened out in the interior domain.

2) Numerous options are available for grid point clustering and surface orthogonality.

Disadvantages :-

1) The computational time required is much larger.

2) The ~~solution~~ matrices must be computed numerically.

Hyperbolic grid generation techniques:-

The elliptic PDE's are used to generate the grid s/m for a closed domain. For open domains, where the outer boundary cannot be prescribed, the hyperbolic techniques can be develop to provide the required grid s/m.

Advantages:-

1) The grid s/m is orthogonal in 2-dimension.
2) This method is computationally must faster than the elliptic technique

Disadvantages:-

1) Extension to 3-D with complete orthogonality is not possible.
2) The boundary discontinuity may be propogated in to the interios domain.
3) This method canot be used for the domains where the outer boundary is specified.

Parabolic grid generators:-

To combine the benefits of the elliptic and hyperbolic s/ms, the parabolic grid generation is

developed.

A parabolic grid generation technique prevents the propogation of boundary discontinuity and are computationally ~~must~~ much faster.

14-7-1

Consistency

A finite difference approximation of a partial differential eqn is consistent if the finit diff eqn approaches the partial diff eqn as the grid size becomes zero.

Stability

A numerical scheme is said to be stable, if any errors introduced in the finit diff eqn doesnot grow with the solution of finite diff eqn.

Convergence

A finite diff scheme is convergent, if the solution of FDE approaches that of the PDE as the grid size becomes zero.

Lax's equivalence theorem

For a FDE which approximates a well-posed, linear initial value problem, the necessary and sufficient condition for convergence is that the finite diff equ must be stable and consistent.

Discretization Error

It is the difference b/w the analytical solution of PDE and the exact solution of the corresponding FDE.

Round-off Error

It is the numerical error introduced after a repetative no. of calculations in which the computer is constantly rounding the numbers to some significant figure.

Consistency Analysis

? Check the consistency for the given model equation using FTCS method.

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}$$

ie $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \right]$$

From Taylor series expansion

$$u_i^{n+1} = u(t+\Delta t) = u(t) + \Delta t \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u}{\partial t^2} + \cdots$$

$$u_{i+1}^n = u(x+\Delta x) = u(x) + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2} + \cdots$$

$$u_{i-1}^n = u(x-\Delta x) = u(x) - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u}{\partial x^2} + \cdots$$

$u_i^n = u(t)$

$$\frac{u(t) + \Delta t \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2} \frac{\partial^2 u}{\partial t^2} - u(t)}{\Delta t} = -a \left[\frac{u(x) + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} \right) - \left(u(x) - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} \right)}{2\Delta x}$$

$$\therefore \frac{\partial u}{\partial t} + \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} = -a \left[\frac{2 \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{6} \frac{\partial^3 u}{\partial x^3}}{} \right]$$

$\Delta t, \Delta x = 0$

$$\Rightarrow \frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}$$

∴ The given eqn is consistent for FTCS method.

? Check the consistency of the given model eqn using CTCS method.

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}$$

Ans ie $$\frac{u_p^{n+1} - u_p^{n-1}}{2\Delta t} = -a \left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \right]$$

$$\frac{\left(u(t)+\Delta t\frac{\partial u}{\partial t}+\frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\right)-\left(u(t)-\Delta t\frac{\partial u}{\partial t}+\frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\right)}{2\Delta t}$$

$$=-a\left[\frac{\left(u(x)+\Delta x\frac{\partial u}{\partial x}+\frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\right)-\left(u(x)-\Delta x\frac{\partial u}{\partial x}+\frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\right)}{2\Delta x}\right]$$

$$\therefore \frac{\partial u}{\partial t}=-a\left[\frac{\partial u}{\partial x}\right]$$

∴ The given eqn is consistent for CTCS method.

? Check the consistency for the given eqn using FTCS method

$$\left[\frac{\partial u}{\partial t}=a\frac{\partial^2 u}{\partial x^2}\right] \rightarrow \text{Parabolic}$$

explicit (FTCS, CTCS) implicit

Ans $$\frac{u_p^{n+1}-u_p^n}{\Delta t}=a\left[\frac{u_{p+1}^n-2u_p^n+u_{p-1}^n}{(\Delta x)^2}\right]$$

$$\text{i.e } \frac{\left(u(t) + \Delta t \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}\right) - u(t)}{\Delta t}$$

$$= a\left[\frac{\left(u(x) + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\right) - 2\left(u(x) + \left(u(x) - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2}\right)\right)}{(\Delta x)^2}\right]$$

$$\frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} = a\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$(\Delta x, \Delta t \to 0)$

$$\therefore \frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

∴ The given eqn is consistent for FTCS method.

Stability Analysis

If the error introduced in the FDE are not controlled then the growth of the error with the solution of FDE will result in an unstable solution.

Understanding and controlling these errors,

Stability Analysis

If the error introduced in the FDE are not controlled, then the growth of error with the Soln of FDE results in an unstable Eqn. Understanding & controlling the error by stability Analysis is essential for the successfull Soln of FDE.

There are two types of stability analysis

1. Discrete perturbation stability Analysis.
2. Von-Neumann.

The second one is the most commonly used..

Von-Neumann Stability Analysis

In this method the Soln of FDE is expanded using Fourier Series and the decay or growth of the Amplification Factor. indicates wheather or not the algorithm is stable.

To illustrate this procedure Assume a Fourier component of u_i^n as shown below.

$$u_i^n = v^n e^{I(p\Delta x)i}$$

where $v^n \rightarrow$ Amplitude

$I = \sqrt{-1}$

$p \Rightarrow$ wave number

$p\Delta x =$ Phase angle.

Similarly,

$$u_{i\pm1}^n = v^n e^{I(p\Delta x)i \pm 1}.$$

$$u_i^{n+1} = u^{n+1} e^{I(p\Delta x)i}$$

$$u_i^n = v^n e^{I\theta_i}$$
$$u_{i\pm1}^n = v^n e^{I\theta(i\pm1)}$$ —(1)

$$v_i^n = v^n e^{I\theta_i}$$
$$v_{i\pm1}^n = v^n e^{I\theta(i\pm1)}$$
$$u_i^{n\pm1} = u^{n\pm1} e^{I\theta_i}$$ —(2)

1 Perform the stability Analysis of the given model Eqn using FTCS method.

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

A) Given eqn is a parbolic PDE & the FTCS scheme is given by

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a\left[\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{(\Delta x)^2}\right]$$

Using eqn (2), the above eqn can be written as

$$u_i^{n+1} = u$$

$$\frac{v^{n+1} e^{I\theta_i} - v^n e^{I\theta_i}}{\Delta t} = a\left[\frac{v^n e^{I\theta(i+1)} + v^n e^{I\theta(i-1)} - 2v^n e^{I\theta_i}}{(\Delta x)^2}\right]$$

$$v^{n+1} e^{I\theta i} = \frac{a\Delta t}{(\Delta x)^2}\left[v^n e^{I\theta(i+1)} + v^n e^{I\theta(i-1)} - 2v^n e^{I\theta i}\right] + v^n e^{I\theta i}$$

$$= \frac{a\Delta t}{(\Delta x)^2}\left[v^n e^{I\theta i} + e^{I\theta} + v^n e^{I\theta i} \cdot e^{I\theta} - 2v^n e^{I\theta i}\right] + v^n e^{I\theta i}$$

$$v^{n+1} = \frac{a\Delta t}{(\Delta x)^2}\left[v^n e^{I\theta} + v^n e^{-I\theta} - 2v^n\right] + v^n$$

where $\frac{a\Delta t}{\Delta x^2} = d \rightarrow$ diffusion factor

$$v^{n+1} = v^n\left[1 + d(e^{I\theta} + e^{-I\theta} - 2)\right]$$

We have,

$$e^{I\theta} = \cos\theta + I\sin\theta$$

$$e^{-I\theta} = \cos\theta - I\sin\theta$$

$$v^{n+1} = v^n\left[1 + d(\cos\theta + I\sin\theta + \cos\theta - I\sin\theta - 2)\right]$$

$$= v^n\left[1 + d(2\cos\theta - 2)\right]$$

$$= v^n\left[1 + 2d(\cos\theta - 1)\right]$$

$$v^{n+1} = v^n\left[1 - 2d(1 - \cos\theta)\right]$$

Now introducing amplification factor 'G' for calculating the stability, we can write.

$G = \frac{v^{n+1}}{v^n}$. in terms of error $G = \frac{\varepsilon^{n+1}}{\varepsilon^n}$.

Therefore here $G = \frac{v^{n+1}}{v^n} = -[1 - 2d(1-\cos\theta)]$

and the condition for Von-Neuman stability analysis is given by $|G| \leq 1$ i.e, $-1 \leq G \leq 1$

with the maximum value of θ the above eqn becomes. ($\theta = 180^\circ$)

$$G = 1 - 2d(1 - (-1))$$

$$G = 1 - 2d(2) = 1 - 4d.$$

For $G \geq -1$

$$1 - 4d \geq -1$$

$$-4d = -2$$

$$-d \geq 1/2 \quad \text{or} \quad d \leq 1/2.$$

For $G \leq 1$

$$1 - 4d \leq 1$$

$$-4d \leq 0$$

$$-d \leq 0$$

$$d \geq 0$$

$$0 \leq d \leq 1/2.$$

For a given equation stable using FTCS method. The condition is that $0 \leq d \leq 1/2$.

Explicit and implicit finite difference method

Explicit and implicit methods are approaches used in the numerical analysis for obtaining the numerical solutions of time dependent ordinary and partial differential eqns.

Explicit methods calculate the state of a system at a later time from the state of the system at the current time. If $Y(t)$ is the current state of the s/m and $Y(t+\Delta t)$ is the state of the s/m at a later time, then

$$Y(t+\Delta t) = F(Y(t))$$

Implicit methods find a solution by solving an eqn involving both the current state of the s/m and the state of the s/m at a later time.

$$G\left[Y(t), Y(t+\Delta t)\right] = 0$$

Explicit scheme

Advantages :-

1) The solution of the algorithm is generally simple.

Disadvantages :-

1) For a given space increment (Δx), the increment in time (Δt) must be less than a specific limit set by the stability condition.

2) This method requires many time steps to carry out the calculation over a given time interval.

Implicit scheme

Advantages :-

1) The stability can be maintained for much larger values of Δt.

2) Less no. of time steps can be used to carry out the analysis over a given time interval.

Disadvantages :-

1) In this method, the procedure involved is much lengthier than the explicit scheme. ~~Since manipul~~

2) Since manipulation is required at each time step, the calculation time is larger compared to explicit approach.

Parabolic partial differential equation

Consider a model eqn,

$$\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} \qquad \text{——— (A)}$$

Explicit methods :-

1) Foreward Time central space (FTCS) :-

In this method, a forward difference

approximation is given to the time derivative and a central diff approximation is given to the space derivative.

Here eqn (A) becomes

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a \left[\frac{u_{i+1}^1 + u_{i-1}^n - 2u_i^n}{(\Delta x)^2} \right]$$

2) Central Time Central space (CTCS):-

In this method central diff approximation is applied to time derivative and space derivative.

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = a \left[\frac{u_{i+1}^n + u_{i-1}^n - 2u_i^n}{(\Delta x)^2} \right]$$

3) Du Fort - Frankel method :-

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = a \left[\frac{u_{i+1}^n - 2\left(\frac{u_i^{n+1} + u_i^{n-1}}{2}\right) + u_{i-1}^n}{(\Delta x)^2} \right]$$

Implicit methods :-

1) Laasonen Method :- (FTCS)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}$$

2) Crank - Nicolson method :-

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{a}{2}\left[\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right]$$

3) Beta - Formulation :-

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = a\left[\beta\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2}\right) + (1-\beta)\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}\right]$$

When $\beta = \frac{1}{2}$, the above expression becomes same as that of Crank-Nicolson method.

When $\beta = 0$, the above expression becomes same as that of explicit FTCS method.

Elliptic partial differential equations

Consider the model eqn,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The above eqn can be analysed by below mentioned

methods.

1) Jacobi Iteration method.
2) Point Gauss-Siedel iteration method.
3) Line Gauss-Siedel iteration method.

Hyperbolic partial differential equations

The model eqn can be written as,

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x}$$

Explicit methods:-

1) Euler's FTCS method :-

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \right]$$

2) Euler's FTFS method:-

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a \left[\frac{u_{i+1}^n - u_i^n}{\Delta x} \right]$$

3) CTCS method:- (Mid point Leap frog method)

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -a \left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \right]$$

Implicit methods :-

1) Eulers BTCS method :-

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} = -a\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}\right]$$

2) FTCS (crank - Nicolson method) method :-

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{-a}{2}\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}\right]$$

Linear and non-linear wave propagation problems are governed by PDE of hyperbolic type. The unsteady Eulers equation governing flow of an inviscid fluid are a system of first order non-linear hyperbolic type equations.

The model equation of hyperbolic type is the first order wave equation.

$$\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x} \longrightarrow (1)$$

Now

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = -c \frac{\partial}{\partial x}\left(-c \frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (2)$$

The above equation is the second order wave eqn.

Explicit FTCS scheme of eq (1)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -c \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

The truncation error is of order $O(\Delta t, \Delta x^2)$.

Explicit FTFS Scheme of eq ①

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -c \frac{u_{i+1}^n - u_i^n}{\Delta x}$$

The truncation error is of order $O[\Delta t, \Delta x]$

Upwind scheme (FTBS) of eq ①

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -c \frac{u_i^n - u_{i-1}^n}{\Delta x}$$

The truncation error is of order $O[\Delta t, \Delta x]$

LAX-WENDROFF TECHNIQUE

The Lax-Wendroff technique is an explicit, finite difference method particularly suited to marching solutions. A good example of a fluid flow field governed by hyperbolic equation is the time-marching solution of an inviscid flow using the unsteady Euler equation.

continuity :- $$\frac{\partial \rho}{\partial t} = -\rho \frac{\partial u}{\partial x} - u \frac{\partial \rho}{\partial x} \longrightarrow (A)$$

x-momentum :- $$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - u \frac{\partial u}{\partial x} \longrightarrow (B)$$

In above equations we have assumed no body force.

The solution can be established using a time-marching approach, the equation has to be arranged with time derivative on left side and spatial derivatives on right side. The Lax-Wendroff method is predicted on a taylor series expansion in time as follows.

Choose any dependent variable, for example consider density, ρ. We can write

$$\rho(t+\Delta t) = \rho(t) + \Delta t \frac{\partial \rho}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 \rho}{\partial t^2} + \dots \longrightarrow (3)$$

The above equation can also be expressed as

$$\rho_i^{n+1} = \rho_i^n + \Delta t \left(\frac{\partial \rho}{\partial t}\right)_i^n + \frac{\Delta t^2}{2} \left(\frac{\partial^2 \rho}{\partial t^2}\right)_i^n + \dots \longrightarrow (4)$$

Here ρ_i^{n+1} is the value of ρ at i and $t+\Delta t$

In eq (4) ρ_i^t is known and Δt is specified. Therefore eq (4) can be used to calculate ρ_i^{n+1} if we have numbers for the derivatives $\left(\frac{\partial \rho}{\partial t}\right)_i^n$ and $\left(\partial^2\rho/\partial t^2\right)_i^n$. The numbers for the derivatives are obtained from governing flow equations (A & B).

From eq (A)

$$\frac{\partial \rho}{\partial t} = -\rho\frac{\partial u}{\partial x} - u\frac{\partial \rho}{\partial x} \longrightarrow (5)$$

Now replacing the space derivatives with central difference scheme as the flow field variable is assumed at $t=0$ (i.e at 'n').

$$\left(\frac{\partial \rho}{\partial t}\right)_i^n = -\rho_i^n\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right] - u_i^n\left[\frac{\rho_{i+1}^n - \rho_{i-1}^n}{2\Delta x}\right] \longrightarrow (6)$$

The above equation gives a number for $\left(\partial\rho/\partial t\right)_i^n$ which can be used in eq (4).

However to complete eq (4) we need expression for $\left(\partial^2\rho/\partial t^2\right)_i^n$.

Now differentiating eq (5) with respect to time

$$\frac{\partial^2 \rho}{\partial t^2} = -\rho \frac{\partial^2 u}{\partial x \partial t} - \frac{\partial u}{\partial x}\frac{\partial \rho}{\partial t} - u\frac{\partial^2 \rho}{\partial x \partial t} - \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial t} \rightarrow (7)$$

Now differentiating eq (5) with respect to x

$$\frac{\partial^2 \rho}{\partial t \partial x} = -\rho \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x} - u\frac{\partial^2 \rho}{\partial x^2} - \frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x} \longrightarrow (8)$$

The procedure is as follows.

(i) In eq (8) replace all derivatives on right side with central difference scheme.

(ii) Insert $\left(\partial^2\rho/\partial t \partial x\right)_i^n$ into eq (7). In eq (7) $\partial u/\partial t$ and $\partial^2 u/\partial x \partial t$ can be obtained from momentum equation

(iii) Insert the number for $\left(\partial^2\rho/\partial t^2\right)_i^n$ into eq (4)

(iv) Insert the number for $\left(\partial\rho/\partial t\right)_i^n$ from eq (6) into eq (4).

(v) Now ρ_i^{n+1} can be calculated.

(vi) Perform the above procedure at every grid point.

The method described above using eq (4), where first and second order derivatives are found by finite difference in space with central difference scheme. and is referred as Lax-Wendroff method.

Mac Cormack's Method.

Mac Cormack's method has been the most popular explicit finite difference method for solving fluid flows. It is related to Lax-Wendroff method but easier to apply. Mac Cormack method like the Lax-Wendroff method is based on a Taylor series expansion in time. Consider the density

$$\rho(t+\Delta t) = \rho(t) + \Delta t \frac{\partial \rho}{\partial t} + \cdots\cdots \quad (1)$$

Considering at grid point 'i' we can write

$$\rho_i^{n+1} = \rho_i^n + \Delta t \left(\frac{\partial \rho}{\partial t}\right)_{avg} \longrightarrow (2)$$

The above equation is a truncated Taylor's series that is of first-order accurate. The derivative $(\partial\rho/\partial t)_{avg}$ is an average time derivative taken between n and n+1. This derivative is evaluated such that the calculation of ρ_i^{n+1} from eq (2) becomes second-order accurate. The average time derivative is evaluated from a predictor-corrector

Step as shown below,

Predictor step

The continuity equation is given by

$$\frac{\partial \rho}{\partial t} = -\rho \frac{\partial u}{\partial x} - u \frac{\partial \rho}{\partial x} \longrightarrow (3)$$

x- momentum equation is

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - u \frac{\partial u}{\partial x} \longrightarrow (4)$$

From eq (3) $\partial \rho / \partial t$ can be calculated at time 'n' using forward difference scheme for space derivatives

$$\left(\frac{\partial \rho}{\partial t}\right)_i^n = -\rho_i^n \left[\frac{u_{i+1}^n - u_i^n}{\Delta x}\right] - u_i^n \left[\frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}\right] \longrightarrow (5)$$

Obtain a predicted value of density $\bar{\rho}_i^{n+1}$, from the first two terms of Taylor series as follows,

$$\bar{\rho}_i^{n+1} = \rho_i^n + \left(\frac{\partial \rho}{\partial t}\right)_i^n \Delta t \longrightarrow (6)$$

In above equation ρ_i^n is known and $\left(\partial \rho / \partial t\right)_i^n$ is a known number from eq (5). Hence $\bar{\rho}_i^{n+1}$ can be easily obtained.

Corrector step

We now will obtain a predicted value of time derivative $\left(\frac{\overline{\partial \rho}}{\partial t}\right)_i^{n+1}$ by substituting the predicted value of $\bar{\rho}_i^{n+1}$ and $\bar{u}_i^{n+1}$. The value of $\bar{u}_i^{n+1}$ can be obtained from momentum equation same as obtained from continuity eqn for $\bar{\rho}_i^{n+1}$.

Now using backward difference in space from eq (3)

$$\left(\frac{\overline{\partial \rho}}{\partial t}\right)_i^{n+1} = -\bar{\rho}_i^{n+1}\left[\frac{\bar{u}_i^{n+1} - \bar{u}_{i-1}^{n+1}}{\Delta x}\right] - \bar{u}_i^{n+1}\left[\frac{\bar{\rho}_i^{n+1} - \bar{\rho}_{i-1}^{n+1}}{\Delta x}\right] \longrightarrow (7)$$

Now calculate the average time derivative as the arithmetic mean between eq (5) and eq (7)

$$\left(\frac{\partial \rho}{\partial t}\right)_{avg} = \frac{1}{2}\left[\left(\frac{\partial \rho}{\partial t}\right)_i^n + \left(\frac{\overline{\partial \rho}}{\partial t}\right)_i^{n+1}\right]$$

Finally the corrected value of ρ_i^{n+1} from eq (2)

$$\rho_i^{n+1} = \rho_i^n + \Delta t \left(\partial \rho / \partial t\right)_{avg}$$

The above predictor - corrector approach is carried out for all grid points.

The MacCormack's technique, a two step predictor-corrector sequence is used with forward difference on the predictor and backward difference on the corrector is a second-order accurate method. Therefore it has the same accuracy as Lax-Wendroff method. The MacCormack method is easier to apply, because there is no need to evaluate the second time derivative as calculated in Lax-Wendroff method.

Central Differencing of incompressible Navier-Stokes equation - The need for staggered Grid

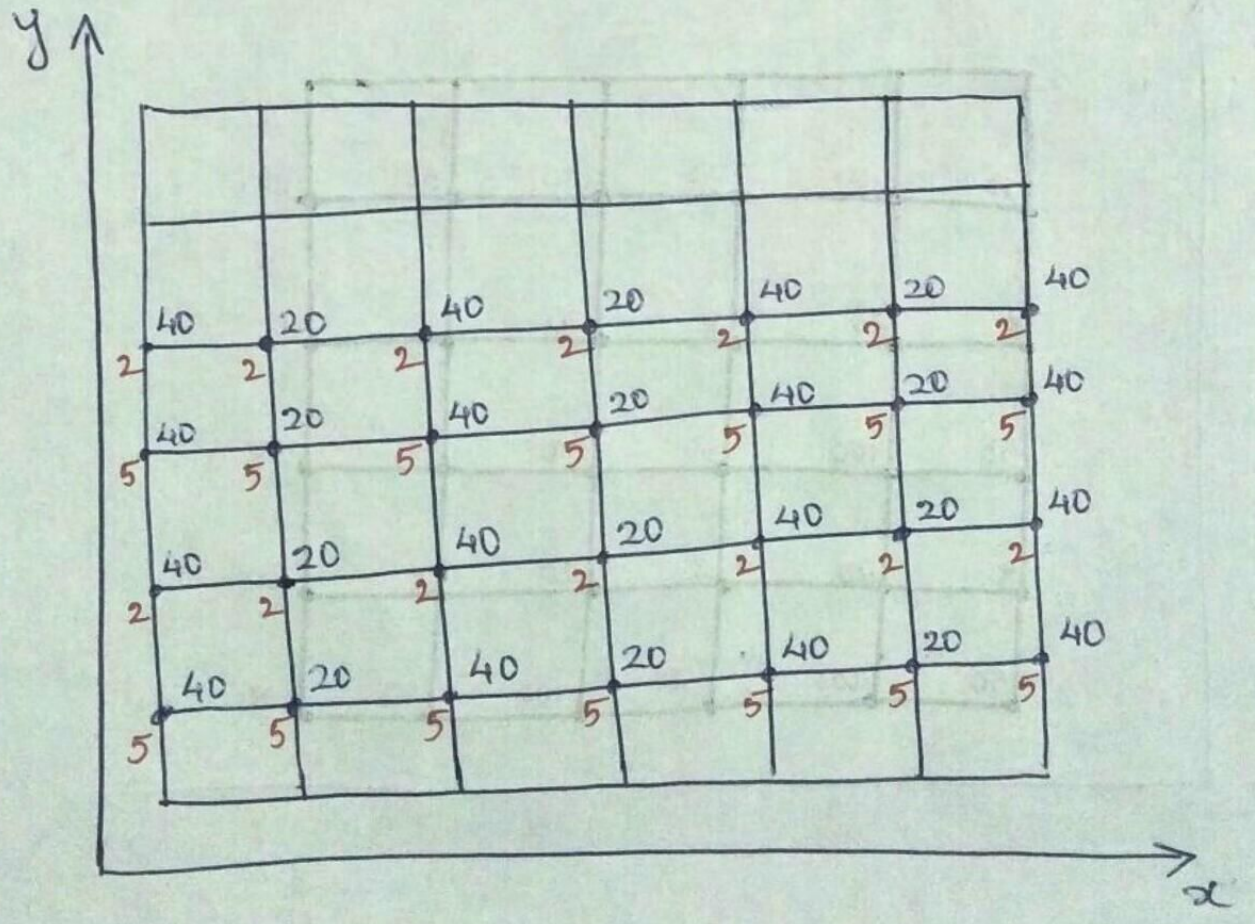
The incompressible continuity eqn in 2D is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \longrightarrow (1)$$

By central differencing scheme

$$\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} = 0 \longrightarrow (2)$$

This difference equation numerically allows the checkboard velocity distribution.

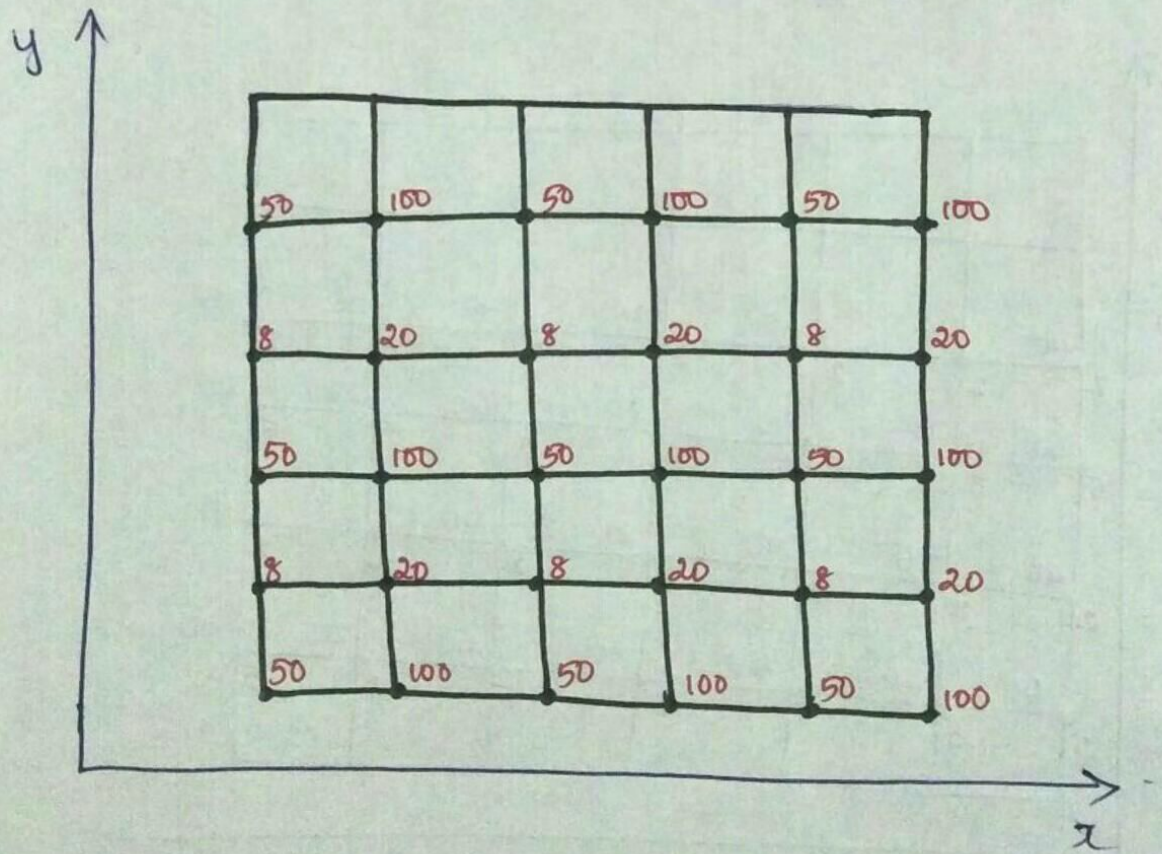


If these numbers are substituted in eq ② both terms are zero at every grid points. But this does not represent any real, physical flow field.

For momentum equation, consider a 2D discrete checkboard pattern as shown

$$\frac{\partial P}{\partial x} = \frac{P_{i+1,j} - P_{i-1,j}}{2\Delta x} \longrightarrow ③$$

$$\frac{\partial P}{\partial y} = \frac{P_{i,j+1} - P_{i,j-1}}{2\Delta y} \longrightarrow ④$$



For the checkboard pattern for pressure distribution eq ③ & ④ will give zero pressure distribution

Two solutions to this case is

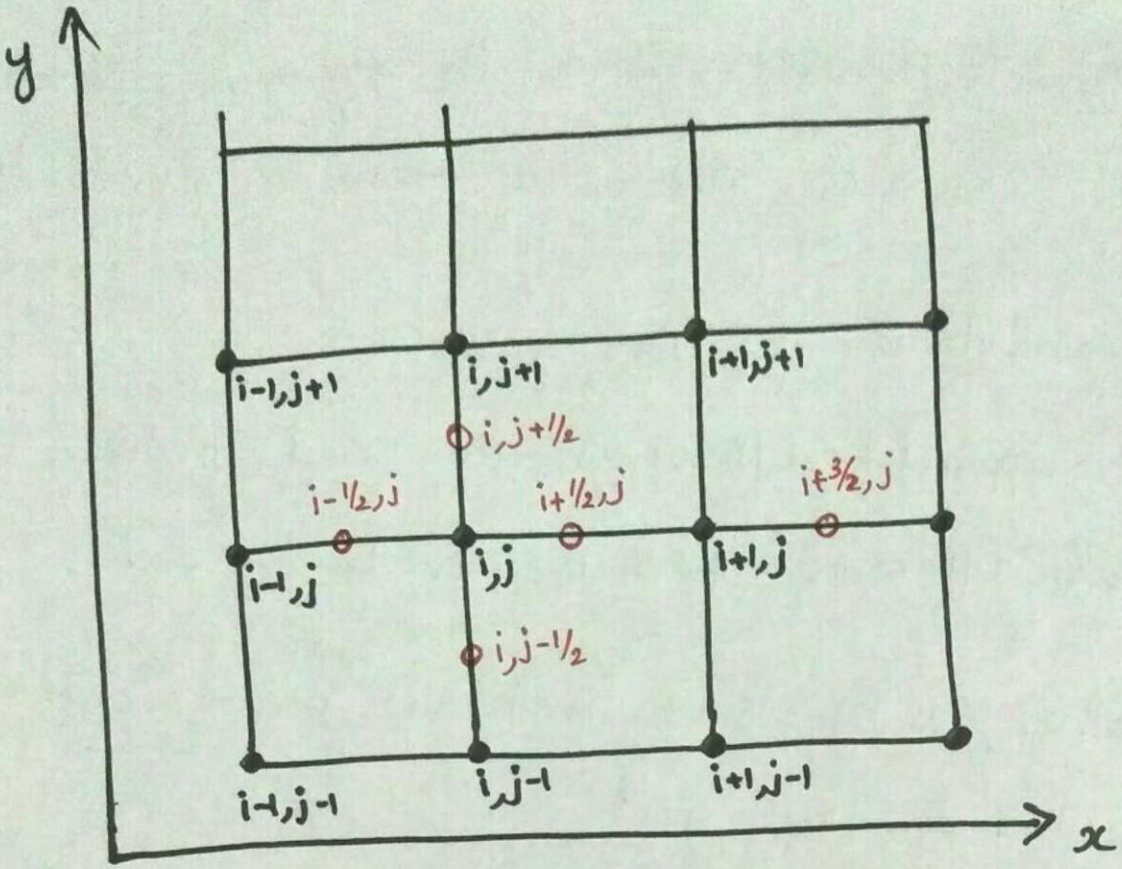
* If upwind differences are used instead of central difference then it can be solved.

* Another way is to maintain central difference but <u>stagger</u> the grid

A staggered grid is shown in figure Here the pressures are calculated at the solid grid points labeled $(i-1,j)$, (i,j), $(i+1,j)$, $(i,j+1)$, $(i,j-1)$ etc. and the velocities are calculated at the open grid points labeled $(i-1/2,j)$, $(i+1/2,j)$, $(i,j+1/2)$, $(i,j-1/2)$ etc.

Specifically 'u' is calculated at $(i-1/2,j)$, $(i+1/2,j)$ etc and 'v' is calculated at $(i,j+1/2)$, $(i,j-1/2)$ etc.

The key feature is that pressure and velocities are calculated at different grid points



An advantage of this staggered grid is that when $u_{i+1/2,j}$ is calculated, a central difference for $\partial p/\partial x$ yields $(P_{i+1,j} - P_{i,j})/\Delta x$ i.e the pressure gradient is based on adjacent pressure points which eliminates the checkboard pattern.

Elliptic Equations

The example of elliptic equation in 2D is as follows

Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow (1)$$

Poisson's equation

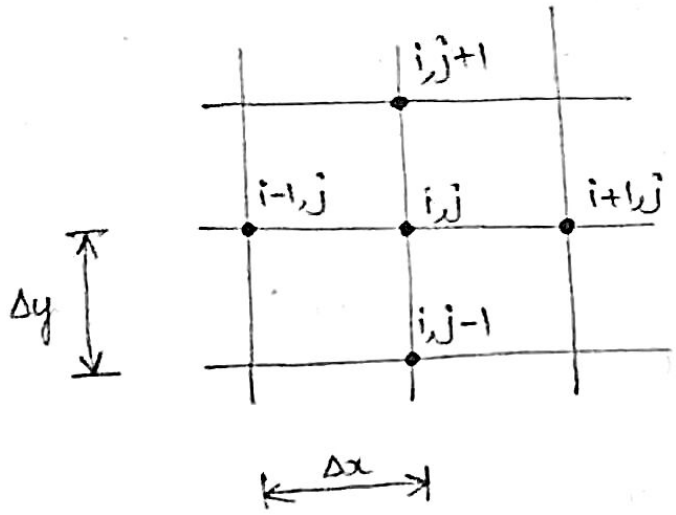
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \longrightarrow (2)$$

From the various finite differening formulations the '<u>five-point formula</u>' is the most commonly used. In this representation of PDE, central-differening which is second order accurate is utilized. Therefore the model equation can be written as

from eq (1)

$$\left[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}\right] + \left[\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2}\right] = 0 \longrightarrow (3)$$

The corresponding grid points are as shown below.



equation (3) can be expressed as follows

$$\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \left(\frac{\Delta x}{\Delta y}\right)^2 \left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) = 0$$

If $\beta = \Delta x / \Delta y$ then

$$u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + \beta^2 u_{i,j+1} - 2u_{i,j}\beta^2 + \beta^2 u_{i,j-1} = 0.$$

$$u_{i+1,j} + u_{i-1,j} + \beta^2 u_{i,j+1} + \beta^2 u_{i,j-1} - 2(1+\beta^2)u_{i,j} = 0 \longrightarrow (4)$$

Similarly equations can be written for all grid points which will give a set of simultaneous equations as eq (4)

Iterative procedures can be easily used to solve these set of equations.

In the iteration procedure, usually an initial solution is guessed and new values are computed. Based on the newly computed values a new solution is obtained. This procedure is repeated ~~until~~ until a specified convergence criteria has been satisfied.

Jacobi Iteration Method.

In this method the dependent variable at each grid point is solved, using initial guessed values of the neighbouring points or previously computed values. Therefore eq ④ is used to compute a new value of $u_{i,j}$ at the new iteration n+1 as follows, from eq ④

$$u_{i,j}^{n+1} = \frac{1}{2(1+\beta^2)} \left[u_{i+1,j}^{n} + u_{i-1,j}^{n} + \beta^2 \left(u_{i,j+1}^{n} + u_{i,j-1}^{n} \right) \right] \longrightarrow (5)$$

where 'n' corresponds to the previously computed values or initial guesses

For example consider a set of equations.

$$\left.\begin{aligned} 7x_1 + 3x_2 + x_3 &= 18 \\ 2x_1 - 9x_2 + 4x_3 &= 12 \\ x_1 - 4x_2 + 12x_3 &= 6. \end{aligned}\right\} \longrightarrow (6)$$

From above set of equations we can write

$$\left.\begin{aligned} x_1 &= \frac{18}{7} - \frac{3x_2}{7} - \frac{x_3}{7} \\ x_2 &= -\frac{12}{9} + \frac{2x_1}{9} + \frac{4x_3}{9} \\ x_3 &= \frac{6}{12} - \frac{x_1}{12} + \frac{4x_2}{12} \end{aligned}\right\} \longrightarrow (7)$$

From eq (7) the initial values of x_1, x_2 & x_3 can be calculated by setting the other parameters to zero in corresponding equations

i.e for x_1, the value of $x_2 = 0$, $x_3 = 0$.

for x_2, $x_1 = 0$, $x_3 = 0$.

for x_3, $x_1 = 0$, $x_2 = 0$.

Then we will get from each equation

$$\left.\begin{aligned} x_1^{(1)} &= 2.571 \\ x_2^{(1)} &= -1.333 \\ x_3^{(1)} &= 0.5 \end{aligned}\right\} \longrightarrow (8)$$

Now using the values of x_1, x_2 & x_3 from eq(8) we can start the next step from eq(7)

$$x_1 = \frac{18}{7} - \frac{3}{7}(-1.333) - \frac{0.5}{7} = \underline{3.071}$$

$$x_2 = \frac{-12}{9} + \frac{2}{9}(2.571) + \frac{4}{9}(0.5) = \underline{-0.540.}$$

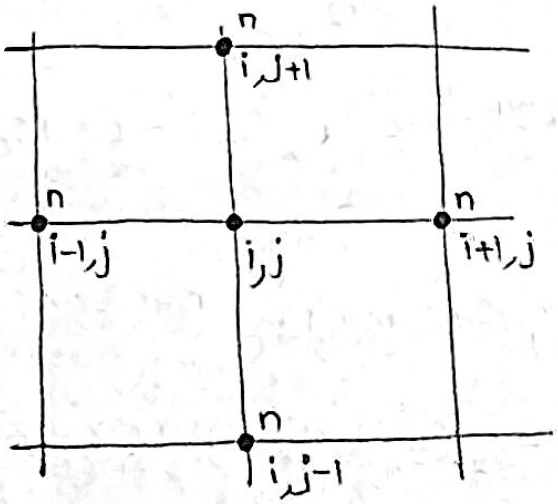
$$x_3 = \frac{6}{12} - \frac{1}{12}(2.571) + \frac{4}{12}(-1.333) = \underline{-0.159}$$

$$x_1^{(2)} = \underline{3.071}$$

$$x_2^{(2)} = \underline{-0.540}$$

$$x_3^{(2)} = -0.159.$$

This process is continued until the desired results are obtained.



Gauss - Seidel Iteration Method.

This method is an extension of Jacobi's iteration method. In this case the initial value at first step is calculated by setting zero to other terms. Then when the value of x_1 is obtained immediatly this value will be used in next calculation of variables x_2 & x_3 in the same step. In other words for each calculation the most current estimate of parameter value is used.

For example from eq (7)

$$x_1 = 18/7 - 3/7(0) - 0/7 = \underline{2.571}$$

$$x_2 = -12/9 + 2/9(2.571) + 4/9(0) = \underline{-0.762}$$

$$x_3 = 6/12 - 2.571/12 + 4/12(-0.762) = \underline{0.033}$$

$$x_1^{(1)} = \underline{2.571}, \quad x_2^{(1)} = \underline{-0.762}, \quad x_3^{(1)} = \underline{0.033}$$

Now

$$x_1^{(2)} = 18/7 - 3/7(-0.762) - 0.033/7 = \underline{2.893}$$

$$x_2^{(2)} = -12/9 + 2/9(2.893) + 4/9(0.033) = \underline{-0.676}$$

$$x_3^{(2)} = 6/12 - 1/12(2.893) + 4/12(-0.676) = \underline{0.034.}$$

Point Gauss - Seidel Iteration Method

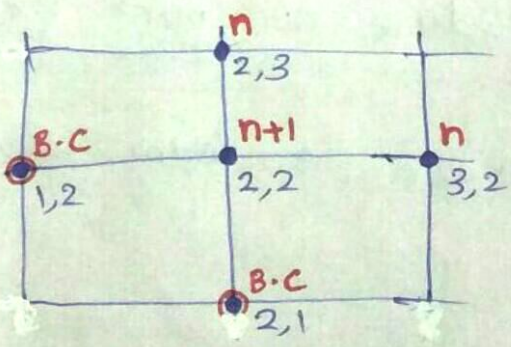
In this method the current values of the dependent variables are used to compute the neighbouring points as soon as they are available. This will increase the convergence rate than Jacobi method.

For example

$$u_{i,j} = \frac{1}{2(1+\beta^2)} \left[u_{i+1,j} + u_{i-1,j} + \beta^2 (u_{i,j+1} + u_{i,j-1}) \right] \longrightarrow (1)$$

The value of $u_{i,j}$ can be calculated at timestep $n+1$ if the values at n for other grid points are available as in explicit method.

$$u^{n+1}_{i,j} = \frac{1}{2(1+\beta^2)} \left[u^{n}_{i+1,j} + u^{n}_{i-1,j} + \beta^2 (u^{n}_{i,j+1} + u^{n}_{i,j-1}) \right] \longrightarrow (2)$$

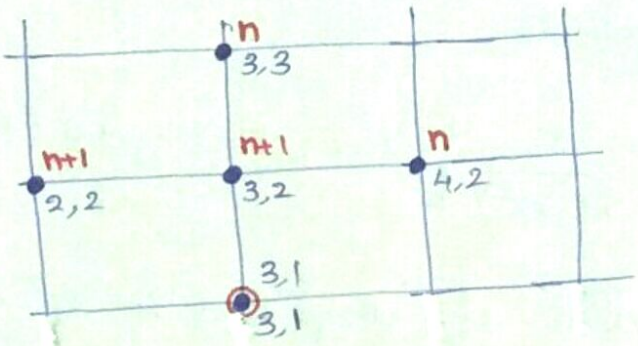


From the grid points shown above

$$u^{n+1}_{2,2} = \frac{1}{2(1+\beta^2)} \left[u^{n}_{3,2} + u_{1,2} + \beta^2 (u_{2,3} + u_{2,1}) \right] \longrightarrow (3)$$

In the eq (3) $u_{2,1}$ and $u_{1,2}$ are provided by the boundary conditions. Only the other two values $u_{3,2}$ and $u_{2,3}$ is obtained from previous step 'n'.

$$\therefore \quad u_{2,2}^{n+1} = \frac{1}{2(1+\beta^2)} \left[u_{3,2}^{n} + u_{1,2} + \beta^2 \left(u_{2,3}^{n} + u_{2,1} \right) \right] \longrightarrow (4)$$



Now from above grid points if $u_{3,2}^{n+1}$ can be written as

$$u_{3,2}^{n+1} = \frac{1}{2(1+\beta^2)} \left[u_{4,2}^{n} + u_{2,2}^{n+1} + \beta^2 \left(u_{3,3}^{n} + u_{3,1} \right) \right] \longrightarrow (5)$$

The above procedure is point iteration method, since only one unknown is being calculated.

The Line Gauss Seidel Iteration Method

In this formulation, three unknowns are calculated in a single line i.e $(i-1,j)$, (i,j), $(i+1,j)$.

The formulation can be expressed as.

(from eq ④ of elliptic eqn representation)

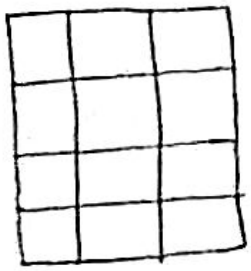
$$u_{i-1,j} - 2(1+\beta^2)u_{i,j} + u_{i+1,j} = -\beta^2 u_{i,j+1} - \beta^2 u_{i,j-1}$$

The above equation when applied to all grid points will result in a system of linear equations.

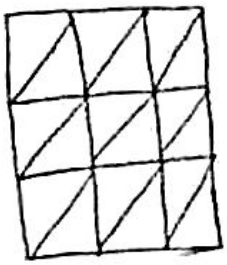
This method converges faster than point Gauss-Seidel method, but it requires more computation time per iteration, since a system of simultaneous equation is being solved. For problems in which the value of dependent variable changes more rapidly in one direction it is advantageous to use the line Gauss-Seidel method in that direction since the solution converges with least number of iterations.

The integral form of equations is discretized, which is the basis of finite volume method. The flow field or domain is subdivided as in finite element method into set of non-overlapping cells that cover the whole domain. In finite volume method (FVM) the term cell is used instead of the term element which is used in FEM. The conservation laws are applied to determine the flow variables in some discrete points of cells called nodes. As in FEM these nodes are at typical locations of the cells such as cell-centres, cell vertices or mid-sides. There is freedom in choosing the cells and nodes. Cells can be triangular, quadrilateral etc. They can be structured or unstructured grid.

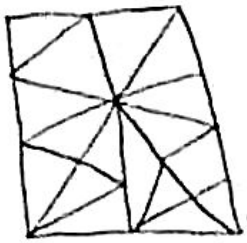
A typical choice of nodes is cell-centres for representing piecewise constant function or cell vertices for representing piecewise linear function.



Structured quadrilateral grid



Structured triangular grid



unstructured triangular grid

Finite volume schemes can be generally categorized into two groups, the first group includes "cell centered" schemes and second group includes "nodal point" scheme

Consider the following model equation

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \longrightarrow (1)$$

The above equation is integrated over an element abcd. Thus we can write

$$\int_{abcd} \left(\frac{\partial Q}{\partial t}\right) dx\,dy = -\int_{abcd} \left(\frac{\partial E}{\partial x} + \frac{\partial F}{\partial y}\right) dx\,dy \longrightarrow (2)$$

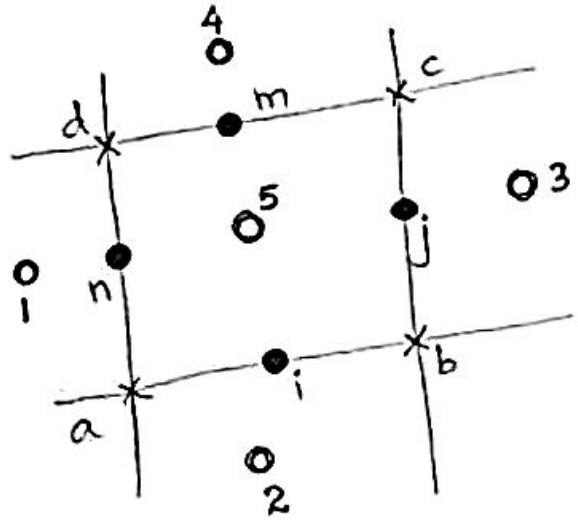
Now Green's theorem is applied to the right hand side of above equation. Greens theorem converts area integrals to line integrals.

$$\int_{abcd} \left(\frac{\partial Q}{\partial t}\right) dx\,dy = -\oint_{abcd} (E\,dy - F\,dx) \longrightarrow (3)$$

Equation (3) is used to develop a cell-centered and nodal point scheme

Cell-centered Scheme.

The integrals in eq (3) are approximated over the element shown in figure



The dependant variable Q is to be solved for node 5 which is located at the center of the element and therefore the scheme is referred as a cell centered scheme. For explicit scheme from eq (3) we can write.

$$\left(\frac{Q_5^{n+1} - Q_5^{n}}{\Delta t}\right) A_{abcd} = -\left(E_i \Delta y_{ab} + E_j \Delta y_{bc} + E_m \Delta y_{cd} + E_n \Delta y_{da}\right) + \left(F_i \Delta x_{ab} + F_j \Delta x_{bc} + F_m \Delta x_{cd} + F_n \Delta x_{da}\right) \longrightarrow (4)$$

where A_{abcd} is the area of the cell and points i, j, m, n represent midpoints of edges ab, bc, cd, da respectively.

Points 1, 2, 3, 4 & 5 are called the control points of the five quadrilaterals. The x & y increments are expressed as follows.

$$\Delta x_{ab} = x_b - x_a, \quad \Delta x_{bc} = x_c - x_b, \quad \Delta x_{cd} = x_d - x_c, \quad \Delta x_{da} = x_a - x_d \rightarrow (5)$$

$$\Delta y_{ab} = y_b - y_a, \quad \Delta y_{bc} = y_c - y_b, \quad \Delta y_{cd} = y_d - y_c, \quad \Delta y_{da} = y_a - y_d \rightarrow (6)$$

The values of functions E and F at the midpoints of the edges can be evaluated by averaging their values from two control points located on the opposite sides of corresponding edges, i.e,

$$\left.\begin{aligned} E_i &= \frac{E_5^* + E_2^*}{2}, \quad E_j = \frac{E_5^* + E_3^*}{2} \\ E_m &= \frac{E_5^* + E_4^*}{2}, \quad E_n = \frac{E_5^* + E_1^*}{2} \end{aligned}\right\} \rightarrow (7)$$

$$\left.\begin{aligned} F_i &= \frac{F_5^* + F_2^*}{2}, \quad F_j = \frac{F_5^* + F_3^*}{2} \\ F_m &= \frac{F_5^* + F_4^*}{2}, \quad F_n = \frac{F_5^* + F_1^*}{2} \end{aligned}\right\} \rightarrow (8)$$

In above equations if the functions denoted by '*' are evaluated at time level 'n', then it is explicit scheme. If the values at are evaluated at 'n+1', then it is

implicit scheme If the element abcd is rectangular in shape then

$$\Delta x_{ab} = -\Delta x_{cd} \quad , \quad \Delta x_{bc} = -\Delta x_{da}$$

$$\Delta y_{ab} = -\Delta y_{cd} \quad , \quad \Delta y_{bc} = -\Delta y_{da}$$

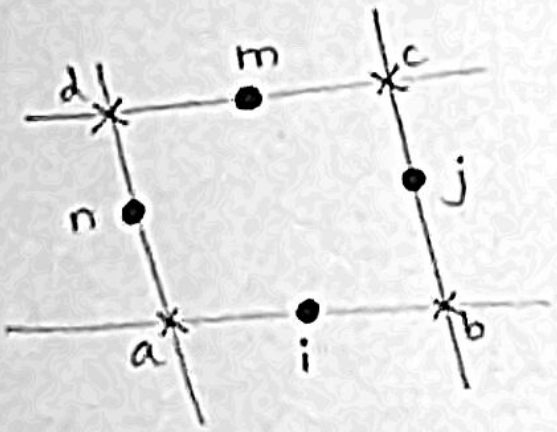
Then eq ④ is equivalent to finite difference formulation where central difference approximation of space derivative is used.

Nodal point scheme

In this approach the dependent variable is evaluated at the vertices of the element. In this case eq ③ can be written as

$$\left[\frac{(Q_a + Q_b + Q_c + Q_d)^{n+1} - (Q_a + Q_b + Q_c + Q_d)^{n}}{+\Delta t}\right] A_{abcd} =$$

$$-(E_i \Delta y_{ab} + E_j \Delta y_{bc} + E_m \Delta y_{cd} + E_n \Delta y_{da}) + (F_i \Delta x_{ab} + F_j \Delta x_{bc} + F_m \Delta x_{cd} + F_n \Delta x_{da}) \longrightarrow (9)$$



where

$$E_i = \frac{E_a^* + E_b^*}{2}, \quad E_j = \frac{E_b^* + E_c^*}{2}$$

$$E_m = \frac{E_c^* + E_d^*}{2}, \quad E_n = \frac{E_d^* + E_a^*}{2} \quad \Bigg\} \rightarrow (10)$$

$$f_i = \frac{f_a^* + f_b^*}{2}, \quad f_j = \frac{f_b^* + f_c^*}{2}$$

$$f_m = \frac{f_c^* + f_d^*}{2}, \quad f_n = \frac{f_d^* + f_a^*}{2} \quad \Bigg\} \rightarrow (11)$$

The above values can be calculated at time level 'n' or 'n+1'. to provide explict & or implicit respectively.

Runge- Kutta Time stepping - Multi stage time stepping

Consider the equation

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial f}{\partial y} = 0. \longrightarrow (1)$$

In integral form over the cell abcd.

$$\int_{abcd} \frac{\partial Q}{\partial t} dx\,dy = -\int_{abcd} \left(\frac{\partial E}{\partial x} + \frac{\partial f}{\partial y}\right) dx\,dy \longrightarrow (2)$$

By applying Green's theorem

$$\int_{abcd} \frac{\partial Q}{\partial t} dx\,dy = -\oint_{abcd} (E\,dy - f\,dx) \longrightarrow (3)$$

From above equation we can write

$$\int_{abcd} \frac{\partial Q}{\partial t}\, dx\,dy = -\Big[E_{i,j-1}\,\Delta y_{ab} + E_{i+1,j}\,\Delta y_{bc} + E_{i,j+1}\,\Delta y_{cd} + E_{i-1,j}\,\Delta y_{da} \Big] + \Big[f_{i,j-1}\,\Delta x_{ab} + f_{i+1,j}\,\Delta x_{bc} + f_{i,j+1}\,\Delta x_{cd} + f_{i-1,j}\,\Delta x_{da} \Big] \longrightarrow (4)$$

In the above equation there is no contribution of the central node (i,j) in the flux balance. This is because the flux balance for a constant flux on a closed surface is zero. Therefore this equation is a similar way of expressing central type finite difference discretization.

The instability of Runge-kutta time stepping can be proved by considering a fourier analysis on a central space discretization of model equation given by

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} \longrightarrow (5)$$

$$\frac{\partial u}{\partial t} = -a \frac{u_{i+1} - u_{i-1}}{2\Delta x}$$

Now substituting $u = Z e^{j\omega x}$

where ω is the wave number and $j = \sqrt{-1}$

$$\frac{\partial}{\partial t} Z e^{j\omega x} = -a \left[\frac{Z e^{j\omega(x+\Delta x)} - Z e^{j\omega(x-\Delta x)}}{2\Delta x} \right]$$

$$z' e^{j\omega x} = -az\left[\frac{e^{j\omega x}\cdot e^{j\omega\Delta x} - e^{j\omega x}\cdot e^{-j\omega\Delta x}}{2\Delta x}\right]$$

$$z' = -az\left[\frac{e^{j\omega\Delta x} - e^{-j\omega\Delta x}}{2\Delta x}\right]$$

when $\theta = \omega\Delta x$ (phase angle)

$$z' = -az\left[\frac{e^{j\theta} - e^{-j\theta}}{2\Delta x}\right]$$

$$z' = -az\left[\frac{\cos\theta + j\sin\theta - (\cos\theta - j\sin\theta)}{2\Delta x}\right]$$

$$z' = -az\left[\frac{2j\sin\theta}{2\Delta x}\right]$$

$$z' = \frac{-azj\sin\theta}{\Delta x}$$

$$z' = \lambda z \qquad \text{where } \lambda = \frac{-aj\sin\theta}{\Delta x}$$

Since the value of λ is on the Imaginary axis the Runge kutta method is unstable. However higher order Runge kutta methods are marginally stable. This

is because it introduces higher order terms in the model equation which results in a real part for the solution.

The higher order Runge kutta time stepping method can be analysed by considering model equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^2 u}{\partial x^2} \longrightarrow (6)$$

$$\frac{\partial u}{\partial t} = -a\left[\frac{u_{i+1}-u_{i-1}}{2\Delta x}\right] + \varepsilon\left[\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta x^2}\right]$$

Since $u = Ze^{j\omega x}$, $u_{i+1} = Ze^{j\omega(x+\Delta x)}$, $u_{i-1} = Ze^{j\omega(x-\Delta x)}$

$$\frac{\partial Ze^{j\omega x}}{\partial t} = -a\left[\frac{Ze^{j\omega(x+\Delta x)} - Ze^{j\omega(x-\Delta x)}}{2\Delta x}\right] + \varepsilon\left[\frac{Ze^{j\omega(x+\Delta x)} - 2Ze^{j\omega x} + Ze^{j\omega(x-\Delta x)}}{\Delta x^2}\right]$$

$$Z' = -Za\left[\frac{e^{j\omega\Delta x} - e^{-j\omega\Delta x}}{2\Delta x}\right] + \varepsilon Z\left[\frac{e^{j\omega\Delta x} - 2 + e^{-j\omega\Delta x}}{\Delta x^2}\right]$$

Since $\theta = \omega\Delta x$

$$Z' = Z\left[-a\left(\frac{e^{j\theta} - e^{-j\theta}}{2\Delta x}\right) + \varepsilon\left(\frac{e^{j\theta} - 2 + e^{-j\theta}}{\Delta x^2}\right)\right]$$

$e^{j\theta} = \cos\theta + j\sin\theta$ & $e^{-j\theta} = \cos\theta - j\sin\theta$

$$Z' = Z\left[-a\left(\frac{2j\sin\theta}{2\Delta x}\right) + \varepsilon\left(\frac{2\cos\theta - 2}{\Delta x^2}\right)\right]$$

$$Z' = Z\left[\frac{-aj\sin\theta}{\Delta x} + \frac{\varepsilon(\cos\theta - 1)}{\Delta x^2}\right]$$

Since the equation now contains a small negative real part, higher order Runge kutta methods now becomes stable.

A modification of the equation (eq ⑥) by adding a fourth-order derivative term instead of a second order derivative term leads to a similar stability effect.

The fourth-order method with simplifications is mostly used since it gives best ratio of allowable time step to compudational work per time step. A simplifed fourth order schome can be written as

$$U^{0}_{i,j} = U^{n}_{i,j}$$

$$U^{1}_{i,j} = U^{0}_{i,j} - \alpha_1 \frac{\Delta t}{\Omega_{i,j}} R^{0}$$

$$U^{2}_{i,j} = U^{0}_{i,j} - \alpha_2 \frac{\Delta t}{\Omega_{i,j}} R^{1}$$

$$U^{3}_{i,j} = U^{0}_{i,j} - \alpha_3 \frac{\Delta t}{\Omega_{i,j}} R^{2}$$

$$U^{4}_{i,j} = U^{0}_{i,j} - \alpha_4 \frac{\Delta t}{\Omega_{i,j}} R^{3}$$

$$U^{n+1}_{i,j} = U^{4}_{i,j}$$

where $\alpha_1 = 1/4$, $\alpha_2 = 1/3$, $\alpha_3 = 1/2$, $\alpha_4 = 1$, R is the Residual

The above simplification has a second order accuracy in time for a non-linear equation. The simplified multi-stage time stepping requires less storage than the classic Runge Kutta time stepping.

Lax - Wendroff Time stepping

Since Lax - Wendroff time - stepping is a very classic explicit time integration method in FDM, the time stepping can be applied to FVM. Consider a model equation

$$\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x} \longrightarrow (1)$$

A Taylor series expansion

$$u(t+\Delta t) = u(t) + \Delta t \frac{\partial u}{\partial t} + \frac{\Delta t^2}{2} \frac{\partial^2 u}{\partial t^2} \longrightarrow (2)$$

$$u^{n+1} = u^n + \Delta t \left(\frac{\partial u}{\partial t}\right)^n + \frac{\Delta t^2}{2} \left(\frac{\partial^2 u}{\partial t^2}\right)^n \longrightarrow (3)$$

The second order derivative $\frac{\partial^2 u}{\partial t^2}$ can be expressed as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)$$

from eq (1) in terms of space derivatives

$$\frac{\partial^2 u}{\partial t^2} = -c \frac{\partial}{\partial x}\left(-c \frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow (4)$$

Now using eq (1) & (4) in eq (3)

$$u^{n+1} = u^n + \Delta t \left(-c \frac{\partial u}{\partial x}\right)^n + \frac{\Delta t^2}{2} \left(c^2 \frac{\partial^2 u}{\partial x^2}\right)^n \longrightarrow (5)$$

In 2D form the equation can be expressed as

$$u^{n+1} = u^n + \Delta t\left(-c\frac{\partial u^n}{\partial x} - c\frac{\partial u^n}{\partial y}\right) + \frac{\Delta t^2}{2}\left(c^2\frac{\partial^2 u^n}{\partial x^2} + c^2\frac{\partial^2 u^n}{\partial y^2}\right) \longrightarrow (6)$$

In FDM the discretization of eq (5) or (6) is called a One-step Lax Wendroff method. In principle a finite volume formulation of eq (5) & (6) is possible since these equations take the form of a flux balance. The fluxes however contains derivatives. Since defining derivatives in FVM is not simple these one-step methods are never used. The most popular two step formulations such as Mac Cormack variant can be used in FVM. Further in one-step formulations the flux balances are lost while solving the equations.

Mac Cormack variant of Lax-Wendroff method

The Maccormack variant can be written as from eq (3)

$$u^{n+1} = \frac{u^n}{2} + \frac{\Delta t}{2}\left(\frac{\partial u}{\partial t}\right)^n + \frac{u^n}{2} + \frac{\Delta t}{2}\left[\frac{\partial}{\partial t}\left(u + \Delta t\frac{\partial u}{\partial t}\right)\right]^n \longrightarrow (7)$$

The predictor can be written as.

$$\bar{u}^{n+1} = u^n + \Delta t\left(\frac{\partial u}{\partial t}\right)^n \longrightarrow (8)$$

Using the predictor eq ⑦ can be expressed as corrector of the form

$$u^{n+1} = \frac{1}{2}\left[u^n + \bar{u}^{n+1} + \Delta t \frac{\partial}{\partial t}\bar{u}^{n+1}\right] \longrightarrow ⑨$$

As explained by MacCormack method the time derivative can be expressed as space derivatives with forward space in predictor step and backward ~~step~~ space in corrector step. Thus MacCormack method is also referred as predictor - corrector method.